STUDY OF MODIFIED STATE-SPACE
METHODS APPLIED TO DISCRETE
TRAFFIC MODELLING

Thesis
Submitted to

Graduate Engineering & Research
School of Engineering
UNIVERSITY OF DAYTON

In Partial Fulfillment of the Requirements for
The Degree
Master of Science in Electrical Engineering

by

David Edwin Champagne
UNIVERSITY OF DAYTON
Dayton, Ohio
August, 1993
STUDY OF MODIFIED STATE-SPACE METHODS APPLIED TO DISCRETE TRAFFIC MODELLING

APPROVED BY:

Frank S. Scarpino, Ph.D.
Associate Professor
Electrical Engineering Department
Committee Chairperson

Malcolm W. Daniels, Ph.D.
Assistant Professor
Electrical Engineering Department
Committee Member

Theresa A. Tuthill, Ph.D.
Assistant Professor
Electrical Engineering Department
Committee Member

Donald L. Moon, Ph.D.
Interim Associate Dean of
Graduate Engineering & Research
School of Engineering

Joseph Lestingi, D. Eng., P.E.
Dean
School of Engineering
ABSTRACT

This paper is an investigation of a possible model for vehicle traffic flow on interstate highways, and related system identification techniques. Data collection is performed by sensors placed on the road surface which detect the passage of vehicles, and grouping these counts over a desired sample interval. Two consecutive sensors provide the input and output for the system. A linear, discrete time state-variable model is developed for vehicle traffic flow which physically relates to the road surface, and is robust in the sense that it can be used to model the non-linear traffic flow. The states of the system are defined as the number of vehicles located within a sub-segment of the distance between sensors. The changing state values over time provide knowledge of the transitional behavior for the vehicles as they move through the system, and can therefore indicate if a slowdown is occurring somewhere within the system. A unique result of the model is that there is a measurable value which is the sum of all the states of the system. A batch least squares system identification technique is derived, for the special case where the system states are available. Correct identification of the system is achieved for the unmatched observer case. A technique to incorporate vehicle velocity into the observer model is introduced. Investigation of observer response to a single fault reveals that all the observer states re-converged to the system states, except for the state were the fault occurred. Observer behavior is found to be improved if a fast sampling rate is used. A fast sample rate forces the state-space quadruple into a highly structured form. Mapping of system parameters between the highway system and an ARMA model provides an exact identification of the system during transitory system behavior, through use of a recursive least squares algorithm applied to the ARMA system.
ACKNOWLEDGMENTS

This paper is dedicated to the newest member of the Champagne family, due to make an appearance early next year, and to it's loving mother, without whom this degree would forever be a goal to be pursued sometime in the future. Thanks for making all the sacrifices, I promise to get a "real" job sometime soon. I would also like to thank my parents for all their support, financial and other, and for always insisting that I really could be an engineer. The research pursued here was initiated with the help of Dr. Scarpino and a group of employees at TRW, and I wish to thank all of them for their support, financial and other, and guidance while pursuing this research. The efforts of my committee members, Dr.'s Tuthill and Daniels, in quickly reading this lengthy paper are greatly appreciated. Thank you Khaled, Taan, and Dave D. for setting me straight about all those simple problems we discussed that I insisted on making difficult. Finally, I wish to thank all the faculty and staff of the University of Dayton Electrical Engineering Department (including Ms. Trudy Krisher), especially Dr. Moon who allowed me to be a part of this happy group, for always making me feel welcome and for guiding me through my two years here.
# TABLE OF CONTENTS

## CHAPTER

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Introduction</td>
<td>1</td>
</tr>
<tr>
<td>I.</td>
<td>Linear, Discrete-Time System Models</td>
</tr>
<tr>
<td>II.</td>
<td>Highway System Model Development</td>
</tr>
<tr>
<td>III.</td>
<td>The State-Space Observer</td>
</tr>
<tr>
<td>IV.</td>
<td>Identification of a System Plant Using a State-Space Observer</td>
</tr>
<tr>
<td>V.</td>
<td>Incorporating Vehicle Velocity Distributions into the System Model.</td>
</tr>
<tr>
<td>VI.</td>
<td>Non-Linearity of the Highway System</td>
</tr>
<tr>
<td>VII.</td>
<td>Case Study of Observer Reaction to a Single Fault</td>
</tr>
<tr>
<td>VIII.</td>
<td>System Identification Using an ARMA Model</td>
</tr>
<tr>
<td>IX.</td>
<td>Conclusions and Future Studies</td>
</tr>
</tbody>
</table>

## APPENDICES

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Appendix A</td>
<td>110</td>
</tr>
</tbody>
</table>

## BIBLIOGRAPHY

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>132</td>
</tr>
</tbody>
</table>
INTRODUCTION

The name "Smart Highway System" loosely encompasses traffic control, global positioning systems, local activity information, and automatic pilot type systems for vehicles traveling interstate highways. As the world's roads become more and more crowded with vehicle traffic a concerted effort is being made to determine ways of observing and ultimately controlling the traffic flow. Control of traffic will help to eliminate slow downs due to rush hour traffic or accidents. A system model is needed which mimics the traffic flow and can be used to determine how to best eliminate any congestion problems. The observation, or sampling of the traffic flow needs to held to a minimum so that the amount of data and also the amount of hardware is held to a minimum. The purpose of the research documented here is to introduce one possible highway traffic flow model based on a linear, discrete-time state-variable system, investigate some of the models properties, and introduce two possible identification techniques.
CHAPTER I
Linear, Discrete-Time System Models

A mathematical model of highway traffic flow is to be developed. This chapter will discuss how an input sequence to a discrete-time system can be related to the output sequence through different system descriptions. The system model which displays the properties required by the highway model is chosen for use in this paper.

A discrete-time system provides a description of how an input data sequence is transformed to an output data sequence. [1, pg. 10] Linear, discrete-time systems can take many forms but all employ adders, multipliers and unit-delay elements. One method of describing a discrete-time system is the difference equation. The difference equation relates the current output sequence value to past output values and current and past values of the input sequence. The past values of both sequences are "stored" in the unit-delay elements and are multiplied by constants. The general form of a linear, constant coefficient difference equation is

\[ y[k] + b_1 y[k-1] + \ldots + b_n y[k-n] = a_0 u[k] + a_1 u[k-1] + \ldots + a_m u[k-m]. \] (1.1)

The difference equation accurately describes the transformation of the input sequence to the output sequence but does not directly provide information about the internal behavior of the system. [1, pg. 11]

More general descriptions of a discrete-time system are the state-variable equations. [1, pg. 70] This model requires more equations than the difference equation model to describe the same system behavior so it is not a minimal realization of a discrete-time system, but does explicitly provide the internal behavior of the system. The states are
defined as the parameters of the system of which one wishes to analyze performance. The state-variable equations employ matrices and vectors. The general form of the state-variable equations are

\[ x[k+1] = Ax[k] + Bu[k] \]
\[ y[k] = Cx[k] + Du[k], \]

where the capitalized variables indicate matrices and the underscore indicates a vector quantity. The state vector is \( x \), the input vector is \( u \), and the output vector is \( y \). The matrices \( A \), \( B \), \( C \) and \( D \) constitute a state-space quadruple and relate the current input and current state values to the next state values and the current output. For a system having \( p \) inputs, \( n \) states and \( m \) outputs the \( A \) matrix is \((n \times n)\), the \( B \) matrix is \((p \times n)\), the \( C \) matrix is \((n \times m)\) and the \( D \) matrix is \((p \times m)\). Figure 1.1 shows a block diagram representing equations 1.2 and 1.3. The states in figure 1.1 are defined as the outputs of the unit-delay element, the \( \Delta \) block, therefore the next value of the states is the input to the unit-delay element.

![Figure 1.1 - Block diagram for the general State-Variable Equations](image-url)
A SISO 2\textsuperscript{nd} order system is described by state-variable equations 1.4 and 1.5 which indicate the specific entries of the state-variable equations. A corresponding block diagram is shown in figure 1.2.

\[
\begin{bmatrix}
  x_1[k+1] \\
  x_2[k+1]
\end{bmatrix} =
\begin{bmatrix}
  a_{11} & a_{12} \\
  a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
  x_1[k] \\
  x_2[k]
\end{bmatrix} +
\begin{bmatrix}
  b_{11} \\
  b_{21}
\end{bmatrix}u[k] \quad (1.4)
\]

\[
y[k] =
\begin{bmatrix}
  c_{11} & c_{12}
\end{bmatrix}
\begin{bmatrix}
  x_1[k] \\
  x_2[k]
\end{bmatrix} +
\begin{bmatrix}
  d_{11}
\end{bmatrix}u[k] \quad (1.5)
\]

![Block diagram of a 2\textsuperscript{nd} order State-Variable System](image)

Figure 1.2 - Block diagram of a 2\textsuperscript{nd} order State-Variable System

The output sequence \( y[k] \) can be computed in two different ways from the state-variable equations in order to find the output value at a specific sample instant, \( k = N \). [1, pg. 74] Each method requires that the initial state values be provided. The first way is to solve equations 1.2 and 1.3 iteratively using the counting index \( k \). By setting \( k = 0, 1, 2, \ldots (N-1) \) in equations 1.2 and 1.3, the state value at each sample time is computed and used in the calculation of the next state value until all \( N \) values have been calculated. In
this way, the output value $y[N]$ can be obtained. The other method is to obtain a closed form solution of equation 1.3. The closed form solution for $x[k]$ can be found by inspection of the iterative calculations for the states of equation 1.2. This closed form solution can then be substituted into equation 1.3 to provide a closed form solution for $y[k]$. Given some initial state vector $x[0]$ for equation 1.2, then

$$x[1] = Ax[0] + Bu[0]$$
$$\vdots$$

which by inspection results in the closed form solution

$$x[k] = A^k x[0] + \sum_{m=0}^{k-1} A^{k-1-m} Bu[m], \quad k \geq 0. \quad (1.6)$$

Substituting equation 1.6 into equation 1.3 results in the closed form solution for the output sequence

$$y[k] = CA^k x[0] + \sum_{m=0}^{k-1} CA^{k-1-m} Bu[m] + Du[k], \quad k \geq 0. \quad (1.7)$$

The first term in equation 1.7 is the system response due to the initial conditions (the natural response), and the last 2 terms provide the system response due to the input sequence (the forced response).

The states of a state-variable model can be chosen arbitrarily to provide information about desired internal parameters of the system under investigation, provided that linear difference equations can be written which describe the relationship between the states. The equations relating the states can then be incorporated into the state space quadruple. It is this flexibility in choosing the states and the availability of the internal
behavior of the system which make the state-variable model ideal for the highway model.

The next chapter discusses the system model used to describe highway traffic flow.
CHAPTER II

Highway System Model Development

As mentioned in the introduction, the first objective of this paper is to describe a method for modeling highway traffic flow and to provide a means of predicting traffic flow patterns, in real time, between data collection points. This chapter will introduce the model which not only relates the input to the output traffic flow but also provides a physical meaning that relates directly to the road surface. In chapter 1 it was shown that a state-variable model provides an internal description, and a thoughtful choice of states will provide a physical meaning for the model. The information contained in the data sequences will be defined, and then examples are given which describe how the state-variable description for a discrete-time system will be used as the system model.

To decide on the states of the model, it is helpful to decide what the model of the highway should show. The model is to be used to determine if and where there is a slowdown in vehicle traffic flow compared to some norm, such as the speed limit, in order to make a decision about how the situation can best be corrected. This means that the model should provide information about traffic flow at all points on the highway. The sampling of the traffic will provide vehicle counts and vehicle velocity. Defining the states as the number of vehicles within a defined section of highway will allow for determination of how the vehicles transition from one section into the next. If it is discovered that the vehicles in one section are not leaving that section at the same rate as expected then a bottleneck has been found. The system model will be developed here by considering the
traffic flow as being linear. The non-linear aspect of the traffic flow will be addressed in a later chapter.

The desired data parameters are vehicle count and vehicle velocity. The data will be collected using loop detectors, which sense when a metal object has passed over them, or some similar sensor placed on the road surface. By placing two such sensors adjacent to each other a velocity can be calculated knowing the time it takes for a vehicle to travel between the two sensors. A discrete data value for the input or output sequences is obtained by counting the number of vehicles which pass over a sensor during an arbitrary sampling interval. Because the sensor continuously detects vehicles, the sampling period can easily be modified to collect vehicle counts over any time period. By collecting the data in this manner a discrete value of the number of vehicles is obtained, and is treated as a value that occurs at the sampling instant. For each data value there is also a velocity distribution for all the vehicles within the sample. In a later chapter it will be shown how velocity measurements can be incorporated into the system model.

Eventually all road surfaces will be monitored by the sensors described above. The obvious desire is to place these sensors as far apart as possible in order to reduce the amount of collected data and to reduce maintenance problems. However, distantly spaced sensors provide less information about the traffic flow and no information about the flow between sensors. For example consider I-71 that connects Cincinnati and Columbus, a distance of 103 miles, as shown in figure 2.1. For the sake of this example, no on or off ramps will be considered. Arbitrarily choosing a between sensor distance of 10 miles requires that 11 sensors be place on the road surface. This in turn results in 10 regions bounded by two consecutive sensors, one providing input data and one providing output data.
In order to discuss this modeling further, some terms are now defined to provide a common language which will be used throughout this paper.

**Control Volume:** That region of roadway that is bounded by any two consecutive sensors.

**Sub-Control Volume:** A sub-division of the control volume, an arbitrary number of which are contained within a control volume, and all are of the same length.

**A State:** The number of vehicles physically located within a sub-control volume at any given sampling instant.

**Control Volume Occupancy:** The total number of vehicles physically located within a control volume at any given sampling instant, and is equal to the sum of all the states.

The use of the above definitions allows the model to relate the sub-control volumes to a specific location on the highway, as well as allowing the states to provide a linear system description of traffic flow through the control volume.

The following example describes how algebraic equations can easily be generated that describe the behavior of traffic through the control volume and how these equations are translated into the state-space form.
**Example 1:**

Consider a two mile section of highway which has no on or off ramps. Let sensors be placed only at the ends of this section, producing a control volume of 2 miles. It is desired that there be two sub-control volumes, each being one mile in length. Figure 2.2 shows how this region of highway is mapped to the control and sub-control volumes.

![Diagram of highway section with sensors and sub-control volumes](image)

**Figure 2.2- Mapping of a highway segment into the model description**

All vehicles passing through this control volume are traveling at exactly 60 MPH, and the sampling interval is 1 minute, which means that each vehicle is traveling at 1 mile/minute or 1 mile/sampling period. Therefore any vehicle that passes over the first sensor during a sampling interval is physically located in sub-control
volume S2 at the next sample instant, and is counted in state \( x_2 \). Any vehicle in state \( x_2 \) will be in state \( x_1 \) during the next sampling instant, and it takes two sampling periods for a vehicle to pass completely through the control volume. The algebraic equations that describe the traffic flow through the states are:

\[
x_2[k+1] = u[k] \tag{2.1}
\]

\[
x_1[k+1] = x_2[k] \tag{2.2}
\]

\[
y[k] = x_1[k] \tag{2.3}
\]

Equation 2.1 is read as the number of vehicles in state \( x_2 \) during the next sampling period is dependent only on the input, \( u[k] \), which is the number of vehicles which crossed the first sensor during the current sample period. Equation 2.2 is read as the number of vehicles in state \( x_1 \), the second 1 mile segment, during the next sample period is dependent only on the vehicles that were in state \( x_2 \) during the current sample period. Equation 2.3 is read as the number of vehicles that pass over the second sensor during a sample period is dependent only on the vehicles that were in state \( x_1 \) during that sample period. Equations 2.1 & 2.2 describe the transitions that occur between the states.

Equations 2.1, 2.2, and 2.3 can be translated into a state-space description. There are three matrices which describe the vehicle behavior: a state transition, or plant matrix, an input matrix (matrices A and B of equation 1.2) and, an output matrix (matrix C of equation 1.3). These three matrices operate on an input vector and a state vector. The state space description for equations 2.1 - 2.3 is shown in equations 2.4 & 2.5.

\[
\begin{bmatrix}
  x_1[k+1] \\
  x_2[k+1]
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1[k] \\
  x_2[k]
\end{bmatrix} +
\begin{bmatrix}
  0 \\
  1
\end{bmatrix}
\begin{bmatrix}
  u[k]
\end{bmatrix} \tag{2.4}
\]

\[
y[k] =
\begin{bmatrix}
  1 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1[k] \\
  x_2[k]
\end{bmatrix}. \tag{2.5}
\]
Equation 2.4 has two different sample periods of the state vector; the current sample period version is multiplied by the state transition matrix. The state transition matrix contains only one non-zero element, entry (1,2), which relates the current sample of x2 to the next sample of x1. If the vehicle speeds were not exactly alike, there would be other non-zero terms in the state transition matrix.

Using this model for the highway system relates a physical location on the highway to the model system. Example 2.1 describes a Single Input/Single Output (SISO) system, i.e., there are no on or off ramps in the 2 mile control volume. If there are ramps located within a desired control volume, the traffic flow on these ramps is treated as extra inputs, and the input matrix is modified to add vehicles to the correct sub-control volume for an on ramp and to subtract vehicles for an off ramp. It is necessary that sensors be placed on all ramps in order to detect those vehicles or they cannot be accounted for in the model. If there were an on ramp and an off ramp located in the second sub-control volume of example 2.1, the equations would now be

\[
\begin{bmatrix}
  x_1[k+1] \\
  x_2[k+1]
\end{bmatrix} =
\begin{bmatrix}
  0 & 1 \\
  0 & 0
\end{bmatrix}
\begin{bmatrix}
  x_1[k] \\
  x_2[k]
\end{bmatrix} +
\begin{bmatrix}
  0 & 0 & 0 \\
  1 & 1 & -1
\end{bmatrix}
\begin{bmatrix}
  u[K] \\
  Onramp[k] \\
  Offramp[k]
\end{bmatrix}
\]

(2.6)

\[
y[k] =
\begin{bmatrix}
  1 & 0 \\
  0 & 1
\end{bmatrix}
\begin{bmatrix}
  x_1[k] \\
  x_2[k]
\end{bmatrix}
\]

(2.7)

Any vehicle entering the highway from the on ramp would be counted by a sensor and added to state x2. Similarly, any vehicle leaving the highway on the exit ramp would be subtracted from state x2.

The modeling technique also provides an easy way to link control volumes in order to provide a complete description of the whole highway system. Consider three
successive control volumes which span 4 sensors. The input to the middle control volume is exactly the output of the first control volume and the output of the second control volume is the input to the third control volume. Extending this to cover the whole interstate system is straightforward.

![Flow diagram](image)

**Figure 2.3: Vehicle Transitions From State x5**

Vehicle velocities are incorporated into the state space description as transition percentages from state to state. Figure 2.3 shows a flow diagram describing the possible vehicle transitions from state x5 in a system with more than 6 states. In figure 2.3, the circles indicate the states of the system and the arrows show the possible transitions that may occur during the next sample instant from one state to the next. P54 indicates the percentage of vehicles which will transition from state x5 to state x4, likewise P53 and P52 indicate transitions from state x5 to states x3 and x2 respectively. P55 is the percentage of vehicles which will remain in state x5 during the next sample instant. Obviously no vehicles are allowed to travel backwards on the road so no transitions can occur to previous states. All of the vehicles, 100 % of them, must transition somewhere during the next sampling instant, therefore the sum of all the percentages for one state must equal 100 %. Similar transitions occur for each state. These constraints and
definitions result in an A matrix of the form of equation 2.8. Example 2.2 describes how velocity relates to the transitions of vehicles from state to state.

\[
A = \begin{bmatrix}
P_{11} & P_{21} & \cdots & \cdots & P_{N1} \\
0 & P_{22} & \cdots & P_{52} & \vdots \\
\vdots & \ddots & P_{33} & P_{53} & \vdots \\
& & P_{44} & P_{54} & \ddots \\
\vdots & & \ddots & \ddots & \ddots \\
0 & \cdots & \cdots & 0 & P_{NN}
\end{bmatrix}
\] (2.8)

Example 2:

Consider a system in which there are 20 vehicles in state x5 during sample time 10, i.e., x5[10] = 20. The sample period is 1 minute, and the sub-control volumes are one mile in length. It is known that 12 of the vehicles are traveling at 60 MPH, 3 vehicles are traveling at 50-55 MPH, 3 vehicles are traveling at 80 MPH, and 2 vehicles are traveling at 125 MPH. In this case, the 3 vehicles traveling at less than 1 mile/minute will likely stay in the same state, therefore P55 = 15 %. The vehicles traveling at exactly 1 mile/minute will always transition into the next state so P54 = 60 %. The vehicles traveling at 80 MPH might skip over state x4 and end up in state x3, therefore P53 = 15 %. The 2 vehicles traveling at > 2 miles/minute will be able to transition 3 states ahead, so P52 = 10 %.

Example 2.2 may be unrealistic in that 2 vehicles probably aren't going to be traveling at 125 MPH for very long, but it does illustrate how vehicle velocity is used in the system model.
The percentages developed for vehicle transitions are directly incorporated into the state space equations. The remainder of this paper considers only SISO systems. Equations 2.4 and 2.5 are in the form

\[
x[k+1] = Ax[k] + Bu[k] \\
y[k] = Cx[k] + Du[k],
\]

(2.9) (2.10)

where A, B, C and D are the state transition, input, output and pass through matrices respectively and constitute a state-space quadruple. The state-space quadruple can be expressed as

\[
\begin{bmatrix}
A & B \\ C & D
\end{bmatrix}.
\]

(2.11)

Equation 2.12 incorporates equation 2.11 and is equivalent to equations 2.9 and 2.10.

\[
\begin{bmatrix}
x[k+1] \\
y[k]
\end{bmatrix} =
\begin{bmatrix}
A & B \\ C & D
\end{bmatrix}
\begin{bmatrix}
x[k] \\
u[k]
\end{bmatrix}.
\]

(2.12)

The first column of the square A matrix relates the transitions from state x1 to the other states. The first column of the row matrix C indicates the percentage of state x1 which leaves the control volume. Therefore, the sum of all the entries in the first column of the A and C matrices, which is the first column of the state-space quadruple, must be exactly one. If this sum is less than one, then some of the vehicles in state x1 have been "lost" and are now unaccounted for, and if the sum is greater than one, then vehicles have been "created". Likewise, the second columns of A and C indicate the transitions of state x2, the third columns indicate the transitions of state x3, etc. As discussed in Example 2.2, vehicles cannot travel backwards on the roadway, therefore all entries below the main diagonal of the A matrix must be exactly zero. A non zero term below the main diagonal would indicate a transition backward toward the input sensor. The entries in the last
column of the state-space quadruple, which is column matrix B and the scalar D (for the SISO system), must also sum to one indicating that all the vehicles which passed over the input sensor are now located in some state of the system or have also crossed the output sensor during the same sample period and are therefore accounted for. The latter case, where there is a non-zero entry in the D matrix, will not be allowed. The sampling rate will be such that no vehicle is allowed to travel completely through the control volume in one sample period. If this was allowed, no information would be available about the internal behavior of the system. Therefore, the D matrix will always contain only zero entries. It will also be required that the only non-zero entry in the column B matrix is a one in the last position, that is, vehicles must enter only the first state. This assumption will result in an error in the actual state values because a vehicle which is traveling fast enough and crosses the sensor at the beginning of the sample period will actually be in the second state. However, the trade off between having a small error in the state values and the simplification in modeling is favorable.

Choice of the sampling rate has a direct effect on the entries in the state space quadruple. For traffic that has a mean velocity of 60 MPH a sampling rate of 1 minute provides a one state transition on average if the sub-control volumes are 1 mile in length. However, if the sampling rate is decreased to 30 seconds the vehicle traveling at the average velocity of 60 MPH will require two sample periods to transition into the next state. Obviously there are an infinite number combinations of sampling rate and sub-control volume length which describe the same traffic flow, therefore knowledge of this choice of sampling rate and sub-control volume length must be known so that the entries of the state space matrices can be understood.

The magnitude of the entries of the plant matrix, A, directly provide information about the transitional behavior of the system. The entries on the main diagonal indicate the percentage of vehicles which remain in the current state. Entries above the main diagonal indicate transitions to the following states. Therefore, an increase in the entries
on the main diagonal indicate that fewer vehicles are leaving that state, and a slow-down may have occurred. If the entries on the main diagonal are exactly 1, then no vehicles are leaving that state. An educated guess at the entries of the state-space quadruple is all that can be made. However, the next chapter describes a technique called an observer which helps to provide information about the internal behavior of the system.
CHAPTER III

The State Space Observer

In the majority of cases where state-variable systems are used to model a process the internal states are not accessible to be measured directly. If an accurate model of the process has been developed, then this model can be used to mimic the actual process and the state values of the model can be updated using error feedback of the measurable states of the process. Generally the output is the only measurable value. This modeling technique is called an observer. Because the observer is a mathematical model, the states of the observer are readily available through calculation. The highway model, by design, does not provide a measurement of the internal states of the system, only a measurement of the final state. To obtain measurements of the internal states would require placing vehicle sensors at the boundaries of every sub-control volume which in turn would eliminate the need for any modeling at all!

The observer state equations are developed from the system state equations. Given a system of the form

\[
\begin{align*}
    x[k+1] &= Ax[k] + Bu[k] \quad (3.1) \\
    y[k] &= Cx[k] , \quad (3.2)
\end{align*}
\]

the observer states are defined as an estimate, \( \hat{x} \), of the system states, \( x \). Figure 3.1 shows a block diagram of the system/observer pair. The input sequence is provided to both the system and the observer. The error term, \( e \), is generated as the difference between the two outputs and is fed back to the observer through an error gain matrix, \( M \). The observer state-variable equations are
\[ \dot{x}[k+1] = A\dot{x}[k] + Bu[k] - MC(x[k] - \dot{x}[k]) \]  
\[ \dot{y}[k] = C\dot{x}[k] \]  

Figure 3.1 - Block Diagram of a System/Observer Pair

In order for the observer states to converge to the system states, the system must be observable. Observability is defined as the ability to determine the states of a system given the measurements, in this case the output values. [2] Observability is assured if the matrix in equation 3.5 is of rank \( n \), where \( n \) is the order of the system.

\[
\psi = \begin{bmatrix}
C^T & A^T C^T & (A^T)^2 C^T & \ldots & (A^T)^{n-1} C^T
\end{bmatrix} \]  

(3.5)
Another useful property is controllability. Controllability is defined as the ability to drive the states of the system to some desired values. \cite{2} If the system is not controllable then not all possible situations can be investigated using the model. Controllability is assured if equation 3.6 is of rank n.

\[ \Theta = [B \mid AB \mid A^2C \mid \ldots \mid A^{n-1}C] \] (3.6)

If the C matrix in equation 3.5 or the B matrix in equation 3.6 contains no non-zero entries, then the system cannot be observed or controlled, respectively. Equation 3.5 implies that if any column of the state-space quadruple has only zero entries, then the system is not observable. Equation 3.6 implies that if any row of the state-space quadruple has only zero entries then the system is not controllable.

The error propagation must form a Bounded Input, Mounded Output (BIBO) system or else there will never be convergence to the system. The error term is defined as

\[ e[k] = x[k] - \hat{x}[k] \] (3.7)
\[ e[k + 1] = x[k + 1] - \hat{x}[k + 1]. \] (3.8)

Using equations 1 and 3, equation 8 can be written as

\[ e[k + 1] = (A + MC)e[k]. \] (3.9)

Therefore the eigenvalues of the \((A+MC)\) matrix must lie within the unit circle for the error term to go to zero, which means that the difference between the system and observer states goes to zero. The only variable in equation 3.9 that is under the control of the designer is the M matrix, therefore the entries in M must be chosen such that the eigenvalues of \((A+MC)\) lie within the unit circle. The rate at which the observer states converge to the system states is controlled by the location of the eigenvalues within the unit circle. The closer the eigenvalues are to the origin, the faster the observer response time. If the observer eigenvalues are located closer to the origin than those of the system,
then the observer will react to a disturbance faster than will the system and the observer states will track the system states.

There is a constraint which is placed on the entries in the M matrix for this model. The error term provides a difference in the number of vehicles between the system and the observer. Just as in the state-space quadruple, all the vehicles which are in the error term must be fed back to the system in order to keep the control volume occupancy levels equal between the system and the observer. In equation 3.3, if the number of vehicles in the system is greater than the number in the observer, then this positive difference needs to be added to the observer. Because the C matrix cannot contain negative numbers, the M matrix entries must all be negative, which means the columns of the square MC matrix must sum to negative one.

A useful property of observers is that they "can also be constructed to provide accurate state estimates of time-varying, deterministic systems - provided the observer response time is chosen to be short, relative to the system time variations" [1, pg. 321] This property will be discussed in chapter 6 where the non-linear behavior of the system is discussed.

The error term of equation 3.7 will approach zero asymptotically as long as the observer is "matched" to the system, that is, as long as the state space quadruples are identical. If they are not matched, then the error term will reach a non-zero steady state value. The magnitude of the steady state error is dependent on how far off the two state-space quadruple entries, which are a function of sampling rate, are and on the feedback matrix entries. An example will be used to investigate this behavior.

The following examples are created in Mathcad and show how the system/observer system can be generated. The first example looks at the effects of changing M on the observer response. The second example looks at the effect of changing the sampling rate on convergence of the observer states. The third example looks at the steady state error
which is produced from having an unmatched observer. The programs used to generate these simulations are included in appendix A.

Example 3.1:

This example shows how convergence is effected by the error feedback matrix.

The traffic flow is idealized in this example in that all the vehicles are traveling at a constant 60 MPH. The length of the control volume is 5 miles and is divided into 5, one mile sub-control volumes, making it a 5th order system. The three parts of this example use different feedback matrices. The sampling interval is 1 minute, which results in all the vehicles transitioning into the next state during a sampling period. All the vehicles which pass over the input sensor at sample time k are considered to be physically in the first state, x5, and the output sequence is exactly state x1, the final state in the system. The observer is matched to the system, therefore, the system and observer state-space quadruples are equal.

The PLANT matrix, A, is defined as: 

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The INPUT matrix, B, is defined as: 

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

The OUTPUT matrix, C, is defined as: 

\[
C = \begin{bmatrix}
1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
It is assumed that the traffic flow has been constant for some time, therefore, there are vehicles in the states of the system at the time the observer is turned on. The observer is given zero initial conditions, while the system has 30 vehicles in each state, initially. The input sequence is shown in figure 3.2. The relative time associated with the x axis is kT, where T is the sampling interval, 1 minute.

The error feedback matrix, M, must be chosen such that the eigenvalues of the (A+MC) matrix lie within the unit circle, and the columns of the MC matrix must sum to -1. The system states are Xsys and the observer states are Xobs.

\[
\text{Figure 3.2 - Input Sequence}
\]

a)

The M matrix entries and resulting eigenvalues are:

\[
M = \begin{bmatrix} -0.2 & -0.2 & -0.2 & -0.2 & -0.2 \\ -0.2 & -0.2 & -0.2 & -0.2 & -0.2 \end{bmatrix}
\]

\[
\text{eigenval}(A + MC) = \begin{bmatrix} 0.518 - 0.594i \\ 0.518 + 0.594i \\ -0.282 - 0.633i \\ -0.282 + 0.633i \\ -0.671 \end{bmatrix}
\]

\[
MC = \begin{bmatrix} -0.2 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \end{bmatrix}
\]
The eigenvalues lie inside the unit circle, and the sum of the first column of the MC matrix is -1. The convergence behavior is shown in figure 3.3 for 50 sample periods after the observer is turned on.

![Graph showing system and observer states](image)

**Figure 3.3 - Error Between System and Observer States**

Figure 3.3 shows that the observer states converge to the system states in approximately 24 sample periods.

b)

Different M values are chosen and the observer states are calculated for the same 50 samples as in part A. The new M values and the resulting eigenvalues are:
The results of the simulation are shown in figure 3.4.

\[
M = \begin{bmatrix}
-0.5 \\
-0.3 \\
-0.1 \\
-0.05 \\
-0.05
\end{bmatrix}
\]

\[
eigenval(A + M \cdot C) = \begin{bmatrix}
-0.545 \\
-0.259 - 0.547i \\
-0.259 + 0.547i \\
0.282 - 0.413i \\
0.282 + 0.413i
\end{bmatrix}
\]

\[
M \cdot C = \begin{bmatrix}
-0.5 & 0 & 0 & 0 & 0 \\
-0.3 & 0 & 0 & 0 & 0 \\
-0.1 & 0 & 0 & 0 & 0 \\
-0.05 & 0 & 0 & 0 & 0 \\
-0.05 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The convergence of the states now requires approximately 14 sample periods.

c)

Different M values are chosen and the observer states are calculated for the same 50 samples as in part A. The new M values and the resulting eigenvalues are:
The results of the simulation are shown in figure 3.5.

In figure 3.5 the observer states do not converge to the system states in the 50 sample periods of the simulation.

The eigenvalues of the three parts of example 3.1 are in different location relative to the origin of the z plane. The eigenvalues of part c are closest to the unit circle and make convergence slowest, while part a has eigenvalues closest to the origin and make
convergence the fastest. The results of example 3.1 match the theoretical behavior of eigenvalue placement.

Example 3.2:

This example investigates the effect of changing the sampling rate used to collect vehicle counts on convergence behavior.

The system description is the same as in example 3.1. Each of the three parts of this example use a different sampling rate. The sampling rate directly effects the entries in the state-space quadruple because the sample-to-sample vehicle transitions are effected. The input sequence is that used in example 3.1. When the sampling interval is decreased, the number of vehicles which pass over the input sensor per minute is held constant. The M matrix entries are all equal.

a)

For the first simulation the sampling rate is $T = 30$ seconds. The input sequence is shown in figure 3.6.

![Figure 3.6 - The Input Sequence](image)
Notice in figure 3.6 that the number of vehicles measured at each sample instant is half that of example 3.1. The input is still defined for 40 minutes. Because all the vehicles are traveling at 60 MPH, each vehicle will require two sample periods to pass through each sub-control volume. Therefore, at each sample instant, half the vehicles will remain in their present state and half will transition to the next state. The state-space quadruple, the M matrix and the resulting eigenvalues are:

\[
A = \begin{bmatrix}
0.5 & 0.5 & 0 & 0 & 0 \\
0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0.5 & 0 & 0 \\
0 & 0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0 & 0.5
\end{bmatrix}
\]

The PLANT matrix, A, is defined as:

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

The INPUT matrix, B, is defined as:

\[
C = \begin{bmatrix}
0.5 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

The OUPUT matrix, C, is defined as:

\[
M = \begin{bmatrix}
-0.4 \\
-0.4 \\
-0.4
\end{bmatrix}
\]

\[
eigenval(A + MC) = \begin{bmatrix}
0.768 - 0.356i \\
0.768 + 0.356i \\
0.327 - 0.363i \\
0.327 + 0.363i \\
0.11
\end{bmatrix}
\]

\[
M \cdot C = \begin{bmatrix}
-0.2 & 0 & 0 & 0 & 0 \\
-0.2 & 0 & 0 & 0 & 0 \\
-0.2 & 0 & 0 & 0 & 0 \\
-0.2 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The convergence of the observer states to the system states is shown in figure 3.7.
The sampling rate is reduced to 6 seconds. The input sequence is shown in figure 3.8, and the values are now one tenth of the values in example 3.1. Now each vehicle will require 10 sample intervals to pass through a sub-control volume, and only one tenth of the vehicles will transition to the next state during a sample period. The length of the input sequence is still 40 minutes.
Figure 3.8 - The Input Sequence

The state-space quadruple, the M matrix and the resulting eigenvalues are:

\[
\begin{bmatrix}
0.9 & 0 & 0 & 0 \\
0 & 0.9 & 0 & 0 \\
0 & 0 & 0.9 & 0 \\
0 & 0 & 0 & 0.9 \\
\end{bmatrix}
\]

The PLANT matrix, A, is defined as:

\[
A = \begin{bmatrix}
0 & 0.9 & 0 & 0 \\
0 & 0 & 0.9 & 0 \\
0 & 0 & 0 & 0.9 \\
\end{bmatrix}
\]

The INPUT matrix, B, is defined as:

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
1 \\
\end{bmatrix}
\]

The OUTPUT matrix, C, is defined as:

\[
C = (0.1, 0, 0, 0)
\]

\[
\begin{bmatrix}
-2 \\
-2 \\
-2 \\
-2 \\
\end{bmatrix}
\]

\[
eigenvalues(A + MC) = \begin{bmatrix}
0.943 - 0.094i \\
0.943 + 0.094i \\
0.761 \\
0.826 - 0.09i \\
0.826 + 0.09i \\
\end{bmatrix}
\]

\[
MC = \begin{bmatrix}
-0.2 & 0 & 0 & 0 & 0 \\
-0.2 & 0 & 0 & 0 & 0 \\
-0.2 & 0 & 0 & 0 & 0 \\
-0.2 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The convergence of the observer states to the system states is shown in figure 3.9.
Figure 3.9 - Error between System and Observer States

Comparison of figures 3.3, 3.7 and 3.9 show that a faster sampling rate requires more samples for the convergence of the observer states to the system states, but a shorter time period. A one minute sampling interval requires 30 minutes to achieve convergence, a 30 second sampling interval requires 20 minutes to achieve convergence, and a 6 second sampling interval requires 10 minutes to achieve convergence. The eigenvalues of the 6 second sample interval simulation of example 3.2b are farthest from the origin, and therefore, require the most samples to achieve convergence.

The past two examples used matched system/observer pairs and the observer states are able to converge, with no error, to the system states. If the observer system is unmatched, then the state-space quadruples are not identical. Example 3.3 investigates the convergence behavior of unmatched system/observer pairs.
Example 3.3:

This example uses three system/observer pairs, each of which are unmatched to a different degree.

The sampling interval is 1 minute for all parts of this example. The input sequence is shown in figure 3.10. It is assumed in all three parts of this example that the traffic is all traveling at 60 MPH, therefore, the observer matrices are exactly those of example 3.1. Vehicles can only enter the first state which means that the input matrix, B, is the same for both the system and the observer.

![Figure 3.10 - The input Sequence](image)

The M matrix is the same as used in example 3.1a for all three parts of this example.

a)

For the first simulation, the vehicles in the system are actually traveling at approximately 54 MPH. This means that one tenth of the vehicles actually remain
in their present state during a sample period. The resulting system plant and output matrices are:

\[
\begin{bmatrix}
0.1 & 0.9 & 0 & 0 & 0 \\
0 & 0.1 & 0.9 & 0 & 0 \\
0 & 0 & 0.1 & 0.9 & 0 \\
0 & 0 & 0 & 0.1 & 0
\end{bmatrix}
\]

\[
A_{sys} = 
\begin{bmatrix}
0 & 0 & 0.1 & 0.9 & 0 \\
0 & 0 & 0 & 0.1 & 0.9 \\
0 & 0 & 0 & 0 & 0.1
\end{bmatrix}
\]

\[
C_{sys} = (0.9, 0, 0, 0, 0)
\]

The calculated system and observer states are shown in figure 3.11.
The graphs in figure 3.11 show how each state of the system and the observer behave over time. Figure 3.12 shows the convergence behavior of the five states.
Comparing figures 3.3 and 3.11, there are more oscillations in the unmatched case than in the matched case, even though the eigenvalues appear to be the same. This is because the true eigenvalues of the observer are now a function of the Asys matrix, not only the Aobs matrix. The error between the states does not go to zero, but reaches a steady state value of 2.5 vehicles in approximately 48 sample periods.

b)

The next simulation has the system vehicles traveling at 45 MPH. Each vehicle requires 1.25 sample periods to transition to the next state, therefore, only 75% of the vehicles will transition to the next state during a sample period. The new plant and output matrices are:
Asys = \[
\begin{bmatrix}
0.25 & 0.75 & 0 & 0 & 0 \\
0 & 0.25 & 0.75 & 0 & 0 \\
0 & 0 & 0.25 & 0.75 & 0 \\
0 & 0 & 0 & 0 & 0.25 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Csys = (0.75 0 0 0 0)

The calculated system and observer states are shown in figure 3.12.
Figure 3.13 - The Five System and Observer States

Figure 3.14 shows the convergence behavior of the five states.

Figure 3.14 - Error between the System and Observer States

The error between the states in figure 3.14 does not go to zero, but reaches a steady state value of 7 vehicles in approximately 49 sample periods.
The final simulation has the system vehicles traveling at 30 MPH. Half of the vehicles will transition during a sample period. The system plant and output matrices are:

$$A_{sys} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \\ 0 & 0 & 0 & 0 & 0.5 \end{bmatrix}$$

$$C_{sys} = \begin{bmatrix} 0.5 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The calculated system and observer states are shown in figure 3.15.
Figure 3.15 - The Five System and Observer States

Figure 3.16 shows the convergence behavior of the five states.

Figure 3.16 - Error between the System and Observer States

The error between the states in figure 3.14 does not go to zero, but reaches a steady state value of 20 vehicles in approximately 49 sample periods.
The steady state error for the produced by the three different systems of example 3.3 are a function of the difference between the system and observer state-space quadruples. The greater the error in the state-space quadruple entries, or the less matched the pair is, the greater the steady state error. The number of samples required to reach the steady state error is essentially the same for all cases.

These three examples have shown that the convergence properties are improved if the $M$ matrix entries are chosen so that the eigenvalues of the observer are close to the origin, if the sampling interval is decreased, and if the observer is matched as closely as possible to the system. Chapter 5 discusses the use of velocity distribution to make the observer state-space quadruple as close to that of the system as possible.
CHAPTER IV
Identification of System Plant Using
a State-Space Observer

In the case of an unmatched observer, as discussed in chapter 3, it was found that the steady state error between the system and observer states was a function of the differences in the entries of the state-space quadruples and the error feedback matrix. A method exists to force convergence of the observer, if the system states are available, through use of a system identification calculation. By first identifying the true system state-space quadruple, then using this quadruple to re-calculate the observer states, convergence will be obtained. This technique will be used in chapter 4, and is presented here so that if a model can be developed which provides convergence of an observer to a system in the unmatched case, identification of the actual system can be obtained. This is discussed at the end of this chapter.

Given a system/observer pair having the following state space description:

\[
\begin{align*}
    x[k+1] &= Ax[k] + Bu[k] \\
    y[k] &= Cx[k] \\
    \text{and} \quad \hat{x}[k+1] &= \hat{A}\hat{x}[k] + \hat{B}u[k] - M(Cx[k] - C\hat{x}[k]),
\end{align*}
\]

the error terms are defined as:
\[ g[k] = x[k] - \hat{x}[k] \] (4.4)
\[ g[k+1] = x[k+1] - \hat{x}[k+1] \] (4.5)
\[ g[k+1] = (A+MC)x[k] - (\hat{A}+\hat{M}\hat{C})\hat{x}[k] + (B-\hat{B})u[k]. \] (4.6)

In the case where the state-space quadruples are matched except for the plant matrices, that is,

\[ A = \hat{A} + \Delta A, \]
\[ B = \hat{B}, \] and
\[ C = \hat{C}, \]

where \( \Delta A \) is defined as the difference between the two plant matrices, equation 4.6 reduces to

\[ g[k+1] = (A + \Delta A + M\hat{C})x[k] - (\hat{A} + M\hat{C})\hat{x}[k] \]
\[ g[k+1] = (\hat{A} + M\hat{C})g[k] + \Delta A x[k]. \] (4.8)

\( \Delta A \) is solved for in equation 4.8 as follows:

\[ \Delta A^*x[k] = g[k+1] - (\hat{A} + M^*\hat{C})^*g[k] \]
\[ x[k]^T\Delta A^T = [g[k+1] - (\hat{A} + M^*\hat{C})^*g[k]]^T \]
\[ \Delta A^T = [x[k]^*x[k]^T]^{-1} * x[k]^*[g[k+1] - (\hat{A} + M^*\hat{C})^*g[k]]^T. \] (4.9)

For an over determined set of solutions for the state vectors of the system and observer, a least squares calculation can be performed using equation 4.9 to solve for \( \Delta A \). The sum of \( \Delta A \) and \( \hat{A} \) will result in the exact system plant, \( A \). From the discussion of equation 2.12, it is known that the sum of the entries in the columns of the state-space quadruple must be one. Because equation 4.9 assumes that there is only a difference in the \( A \) matrices of the system and observer, the \( B \) & \( C \) matrices are matched, then any
differences in the C matrix will appear in $\Delta A$. The differences in the C matrix will be manifested as non-zero values below the main diagonal of the $\Delta A$ matrix, indicating backward transitions of vehicles. Because all the vehicles must be accounted for, the identification assumes that because the observer C matrix is fixed and therefore only that number of vehicles exit the system, the excess vehicles within a state must be sent backward to previous states by placing non-zero entries below the main diagonal. Therefore, the sum of identified entries below the main diagonal is the difference in the C matrices of the system and observer, and should be subtracted from the observer C matrix.

Example 4.1 shows how the use of equation 4.9 results in identification of the system plant. In example 4.1, $X_{sys_0}$ and $X_{sys_1}$ are sets of 10 system state vectors, $x[k]$, such that

$$X_{sys_0} = [x[0] \ x[1] \ \cdots \ x[9]], \quad (4.10)$$

and

$$X_{sys_1} = [x[1] \ x[2] \ \cdots \ x[10]]. \quad (4.11)$$

$X_{obs_0}$ and $X_{obs_1}$ are similar sets of observer state vectors. These matrices provide 10 solutions for the 5th order system.
Example 1:

This example utilizes the results of example 3.3c. The difference between the system and observer state-space quadruples is calculated. In order to perform a least squares calculation, there must be more results than unknowns. For this example, there are 5 states, therefore, there needs to be more than 5 results to have an overdetermined set of results. Ten samples of the states of the system and observer are taken, starting at the first sample.

CALCULATION OF THE OVERDETERMINED SOLUTION SETS:
10 states are selected.

\[
k = 0..9
\]

\[
X_{sys_0}^{<k>} = X_{sys}^{<k>}
X_{obs_0}^{<k>} = X_{obs}^{<k>}
X_{sys_1}^{<k>} = X_{sys}^{<k+1>}
X_{obs_1}^{<k>} = X_{obs}^{<k+1>}
\]

Equation 4.9 is used,

\[
\Delta A = \left( X_{sys_0} X_{sys_0}^T \right)^{-1} X_{sys_0} \left( \left( X_{sys_1} - X_{obs_1} \right) - \left( A_{obs} + M_{obs} \right) \left( X_{sys_0} - X_{obs_0} \right) \right)^T
\]

the results are:

\[
\Delta A^T =
\begin{bmatrix}
0.6 & -0.5 & 1.62 \times 10^{-13} & -5.232 \times 10^{-14} & 0 \\
0.1 & 0.5 & -0.5 & -2.758 \times 10^{-14} & 1.678 \times 10^{-14} \\
0.1 & -3.754 \times 10^{-13} & 0.5 & -0.5 & 0 \\
0.1 & 8.471 \times 10^{-14} & -7.505 \times 10^{-14} & 0.5 & -0.5 \\
0.1 & -1.723 \times 10^{-12} & 8.74 \times 10^{-13} & -2.62 \times 10^{-13} & 0.5
\end{bmatrix}
\]

The calculated system plant is:

\[
\Delta A^T + A_{obs} =
\begin{bmatrix}
0.6 & 0.5 & 1.62 \times 10^{-13} & -5.232 \times 10^{-14} & 0 \\
0.1 & 0.5 & 0.5 & -2.758 \times 10^{-14} & 1.678 \times 10^{-14} \\
0.1 & -3.754 \times 10^{-13} & 0.5 & 0.5 & 0 \\
0.1 & 8.471 \times 10^{-14} & -7.505 \times 10^{-14} & 0.5 & 0.5 \\
0.1 & -1.723 \times 10^{-12} & 8.74 \times 10^{-13} & -2.62 \times 10^{-13} & 0.5
\end{bmatrix}
\]
The first column of the identified plant matrix has non-zero entries below the main diagonal, therefore, the sum of these entries is subtracted from the Cobs matrix.

The identified results are introduced into the observer state update equations, and the states are re-calculated.

\[
A_{obs} = \begin{bmatrix}
0.6 & 0.5 & 0 & 0 \\
0 & 0.5 & 0 & 0 \\
0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0.5
\end{bmatrix}
\]

\[
C_{obs} = (0.4, 0, 0, 0)
\]

\[
k = 0..79
\]

\[
X_{sys}^{k+1} := A_{sys} X_{sys}^k + B \cdot u_k
\]

\[
Y_{sys}^k = C_{sys} \cdot X_{sys}^k
\]

\[
X_{obs}^{k+1} := A_{obs} \cdot X_{obs}^k + B \cdot u_k - M \cdot (C_{sys} \cdot X_{sys}^k - C_{obs} \cdot X_{obs}^k)
\]

The identification is performed again:

\[
k := 0..9
\]

\[
X_{sys_0}^k := X_{sys}^k \\
X_{sys_1}^k := X_{sys}^{k+1}
\]

\[
X_{obs_0}^k := X_{obs}^k \\
X_{obs_1}^k := X_{obs}^{k+1}
\]

\[
\Delta A = (X_{sys_0} X_{sys_0}^T)^{-1} \cdot X_{sys_0} \cdot ((X_{sys_1} - X_{obs_1}) - (A_{obs} + M \cdot C_{obs}) \cdot (X_{sys_0} - X_{obs_0}))^T
\]

The results of the re-calculation are:

\[
\Delta A^T = \begin{bmatrix}
-0.12 & 2.688 \times 10^{-13} & -1.481 \times 10^{-13} & 4.954 \times 10^{-14} & -2.193 \times 10^{-15} \\
-0.02 & 4.27 \times 10^{-14} & -2.27 \times 10^{-14} & 7.244 \times 10^{-15} & 0 \\
-0.02 & 4.523 \times 10^{-14} & -2.49 \times 10^{-14} & 8.091 \times 10^{-15} & 0 \\
-0.02 & 4.63 \times 10^{-14} & -2.523 \times 10^{-14} & 8.16 \times 10^{-15} & 0 \\
-0.02 & 4.459 \times 10^{-14} & -2.491 \times 10^{-14} & 8.209 \times 10^{-15} & 0
\end{bmatrix}
\]
The calculated system plant is:

\[
\Delta A^T + A_{\text{obs}} = \\
\begin{bmatrix}
0.48 & 0.5 & -1.481 \times 10^{-13} & 4.954 \times 10^{-14} & -2.193 \times 10^{-15} \\
-0.02 & 0.5 & 0.5 & 7.244 \times 10^{-15} & 0 \\
-0.02 & 4.523 \times 10^{-14} & 0.5 & 0.5 & 0 \\
-0.02 & 4.63 \times 10^{-14} & -2.523 \times 10^{-14} & 0.5 & 0.5 \\
-0.02 & 4.459 \times 10^{-14} & -2.491 \times 10^{-14} & 8.209 \times 10^{-15} & 0.5
\end{bmatrix}
\]

Now the observer output matrix entry (1,1) will be changed to .52, which is \([1 - A(1,1)]\). The observer matrices are updated again:

\[
A_{\text{obs}} = \\
\begin{bmatrix}
48 & 0.5 & 0 & 0 & 0 \\
0 & 0.5 & 0.5 & 0 \\
0 & 0 & 0.5 & 0.5 \\
0 & 0 & 0 & 0.5
\end{bmatrix}
\]

\[
C_{\text{obs}} = (.52, 0, 0, 0, 0)
\]

The whole process is repeated until all the terms in the \(\Delta A\) matrix are smaller than some desired error, then the iteration is stopped. The identified system state-space quadruple is then used for the observer for calculations involving future inputs.
Similar simulations were performed for a wide variation between the A and C matrices of the system and observer. When only the A matrix differed, use of equation 4.9 resulted in the exact identification of ΔA in one iteration. As in example 4.1, when both the A and C matrices differed, a few iterations of equation 4.9 were necessary to obtain the exact identification. The use of equation 4.9 is only necessary when a difference is seen between the system and observer outputs. When an error is detected, the iteration shown in example 4.1 can hopefully be performed faster than the sampling rate, and will therefore provide an updated observer before another data value is taken.

As mentioned at the beginning of this chapter, this identification requires knowledge of the system states. If a model can be created which forces convergence to the system states during transitory behavior, then an observer can be used in conjunction with this models states to calculate ΔA through use of equation 4.9. Such a model was not found during this research project.
CHAPTER V
Incorporating Vehicle Velocity Distributions
Into the System Model

In the previous chapters it was shown that the steady state error between the observer states and the system states is dependent on how close the entries of the two systems state-space quadruples are to each other, i.e. the error between the two. Therefore, the more information that can be used to match the observer to the model the better the convergence will be. All of the entries in the state-space quadruple are dependent on sampling rate and vehicle velocity. Input velocity distributions can be used to generate transition probabilities within the A and B matrices. Output velocity distributions provide transition information for the A and C matrices. This chapter develops a method to directly map the sample input velocity distribution into the state-space quadruple. A projection of the states will be made, and then a least squares calculation, equation 4.9, will be used to determine a plant for the observer.

In order to facilitate the mapping of the velocity distribution into the state-space quadruple, some standardization of the sampling process is presented here. Sub-control volumes should be set at a given length, L, and sensors should then be placed to accommodate some whole number of sub-control volumes, N. N may vary from sub-control volume to sub-control volume. Data will be collected into data packets over a very short time period, Δt, which should be some division of 1 minute, either 5 or 6 seconds. Actual data values used in the input and output sequences can then be created from a combination of these data packets. In this way, the data sequence sample period,
T, can be matched to the mean velocity to cause a desired transition of the vehicle traveling at the mean velocity from state to state. The vehicle traveling at the mean velocity will be denoted as the "mean vehicle" during the following discussion. If the sampling period is very short most of the vehicles will remain in their current state during the next sample instant, and by increasing the sampling period more and more vehicles will transition to the next state during a sample instant.

Over a windowed set of the data sequence a grand mean, V, can be calculated. Knowing L, the sub-control volume length, will then allow calculation of T so that the mean vehicle transition characteristics are as desired. The formula used is

\[ \alpha V = \frac{L}{T}, \]  

(5.1)  

\[ T \leq \frac{L}{\alpha V}, \text{ for } \alpha = 1, 2, 3 \ldots \]  

(5.2)

where T is chosen as

\[ T = n\Delta t, \text{ for } n = 1, 2, 3, \ldots \]

for the largest possible n. Due to the collection of data over the interval \( \Delta t \) the sequence sample period must be chosen as a whole number of \( \Delta t \). The variable \( \alpha \) describes the transition behavior of the mean vehicle. In general, the mean vehicle will transition to the next state in \( \alpha \) sample periods. For \( \alpha = 1 \), the mean vehicle will always transition to the next state in during a sample period. For \( \alpha = 2 \), the mean vehicle will transition one state every 2 sample periods. Therefore the choice of \( \alpha \) determines the behavior of the mean vehicle.

The calculated mean velocity is found from

\[ \Omega = \frac{L}{\alpha T}, \]  

(5.3)

where \( \Omega \) is greater than or equal to V. \( \Omega \) is superimposed over the velocity distribution of each input sample. The velocity distribution is divided into three regions, R1, R2, and R3, the boundaries of which are dependent on \( \Omega \). Each region corresponds to a different
transitional behavior of the vehicles in each region relative to the mean velocity \( \Omega \). The boundaries are defined as

\[
\begin{align*}
R_1 &= 0 \text{ MPH} < v < \frac{3\Omega}{4} \text{ MPH} \\
R_2 &= \frac{3\Omega}{4} < v < \frac{5\Omega}{4} \text{ MPH} \\
R_3 &= v \geq \frac{5\Omega}{4} \text{ MPH}.
\end{align*}
\]

The percentage of the number of vehicles in each region of the total sample are expressed as \( \tau_1 \), \( \tau_2 \), and \( \tau_3 \), corresponding to the three regions defined above. The transition behavior for the vehicles in each region is defined as:

\( \tau_1 \) % of the vehicles in the sample will require \((\alpha+1)\) sample periods to transition to the next state, then they will require \( \alpha \) sample periods to transition, then \((\alpha+1)\), then \( \alpha \), etc.

\( \tau_2 \) % of the vehicles in the sample will require \( \alpha \) sample periods to transition to the next state.

\( \tau_3 \) % of the vehicles in the sample will require \( \alpha \) sample periods to transition to the next state, then they will require \((\alpha-1)\) sample periods to transition to the next state, then \( \alpha \), then \((\alpha-1)\), etc.

If \( \alpha = 1 \), then \( \tau_3 \) % of the vehicles will skip a state every other sample period. Example 5.1 describes how the 3 regions are determined.

**Example 5.1:**

Consider one sample of the input data sequence to a control volume. The sub-control volume length has been set at \( L = .5 \) miles, and there are 7 sub-control volumes within the control volume. A grand mean has been calculated, \( V = 55.7 \) MPH, using the past 40 samples of the input data sequence. It is desired that the vehicle traveling at the mean velocity should transition into the next state in two sample periods, i.e. \( \alpha = 2 \). The sampling has been done using \( \Delta t = 5 \) seconds. The data sequence sampling interval used was found by:
which leads to

$$\Omega = (0.5 \text{ miles})(3600 \text{ sec/hour}) / (2)(15 \text{ sec.}) = 60 \text{ MPH}$$

The three velocity regions are

- R1 = 0 to 45 MPH
- R2 = 45.1 to 75 MPH
- R3 = greater than 75 MPH

The sample velocity distribution is shown in figure 5.1 along with the three velocity regions.

**Figure 5.1 - Typical Velocity Distribution of One Sample**

Figure 5.1 displays how the regions allow for averaging of the vehicle speeds within the R2 region which are defined to transition every $\alpha$ sample periods. The vehicles traveling at the slow end of the region may not actually transition to the next state while the vehicles traveling near the high end may actually skip some states. However, the sequence samples immediately on either side will have the same behavior and smear into this sample.
One way to map the velocity distributions into the state-space quadruple is to consider each sample individually as it passes through the control volume. Therefore, at each sample instant, the state update equation will be dependent on values calculated from each individual sample of the data sequence. The actual mapping of this behavior into the state-space quadruple is best illustrated through an example. Example 5.2 takes one sample of the input data sequence and tracks the vehicles through the control volume.

**Example 5.2:**

The control volume under consideration has 5 sub-control volumes, each being .5 miles in length. The 12th sample of the input data sequence is considered. This sample has 10 vehicles, \( u[12] = 10 \), and the velocity distribution is such that there is one vehicle each in the velocity regions R1 and R3 and the remaining eight vehicles are in region R2. The corresponding vehicle percentages are; \( \tau_1 = .1, \tau_2 = .8 \) and \( \tau_3 = .1 \). The sampling period \( T \) has been chosen such that the vehicle traveling at the mean velocity will transition to the next state in one sample period (\( \alpha = 1 \)). The initial state values due to this sample are zero. The counting index, \( c \), is started at zero but is actually an offset from the sample within the data sequence. That is, the sample under consideration is the 12th value in the input data sequence, then the state value at time \( k = 12 \) corresponds to \( c = 0 \) in this example. The state values are given below, starting with \( c = 0 \).

\[
\begin{align*}
  x_5[12] &= 0 & x_5[13] &= 10 \\
  x_4[12] &= 0 & x_4[13] &= 0 \\
  x_3[12] &= 0 & x_3[13] &= 0 \\
  x_2[12] &= 0 & x_2[13] &= 0 \\
  x_1[12] &= 0 & x_1[13] &= 0 \\
  x_5[14] &= 1 & x_5[15] &= 0 \text{ from now on} \\
  x_3[14] &= 0 & x_3[15] &= 8 \\
  x_2[14] &= 0 & x_2[15] &= 1 \\
  x_1[14] &= 0 & x_1[15] &= 0
\end{align*}
\]
At sample time $k = 21$, all the vehicles have left the control volume which entered during sample $u[12]$. The behavior of the vehicles in region R1 is, as described earlier, that they are counted in state $x_5$ at $k=13$ and it takes $\alpha+1 = 2$ sample periods to transition to the next state, so they don't transition until $k=15$. The next transition occurs in $\alpha$ periods, so they transition again at $k=16$. Now the cycle is repeated. It requires 2 sample periods, then 1, then 2, etc. to transition. The vehicles in region R2 always transition during the next sample period. The vehicles in region R3 will transition to the next state, then skip a state, then transition, then skip, etc. The above state values can be related to the sample value, $u[12]$, and the vehicle percentages within each region are as follows:

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0 from now on</td>
<td>1</td>
<td>1</td>
<td>8</td>
<td>0</td>
<td>1</td>
<td>8</td>
</tr>
</tbody>
</table>

$x_3[18] = 0$ and from now on

$x_3[19] = 0$

$x_2[18] = 1$

$x_2[19] = 0$

$x_1[18] = 0$

$x_1[19] = 1$

$x_2[20] = 0$

$x_1[21] = 0$

$x_5[12] = 0$

$x_5[13] = 10 = 1* u[12]$

$x_4[12] = 0$

$x_4[13] = 0$

$\vdots$

$x_1[12] = 0$

$x_1[13] = 0$

$x_5[14] = 1 = \tau_1u[12]$

$x_5[15] = 0$

$x_4[14] = 9 = (\tau_2 + \tau_3)u[12]$

$x_4[15] = 1 = \tau_1u[12]$

$x_3[14] = 0$

$x_3[15] = 8 = \tau_2u[12]$

$x_2[14] = 0$

$x_2[15] = 1 = \tau_3u[12]$

$x_1[14] = 0$

$x_1[15] = 0$

$x_3[16] = 1 = \tau_1u[12]$

$x_3[17] = 1 = \tau_1u[12]$

$x_2[16] = 8 = \tau_2u[12]$

$x_2[17] = 0$

$x_1[16] = 1 = \tau_3u[12]$

$x_1[17] = 8 = \tau_2u[12]$
\[ x3[18] = 0 \quad \text{x3[19] = 0 and from now on} \]
\[ x2[18] = 1 = \tau_1 u[12] \quad \text{x2[19] = 0} \]
\[ x1[18] = 0 \quad \text{x1[19] = 1 = \tau_1 u[12]} \]

\[ x1[20] = 1 = \tau_1 u[12] \quad \text{x1[21] = 0} \]

The vehicle percentages can be mapped into a time varying column matrix which relates the input sample value to the state values. The state vector subscript indicates the data sample under consideration, and the counting index is then the offset from the subscript.

\[
\begin{align*}
x_{12}[1] &= [0 \ 0 \ 0 \ 0 \ 1]^T u[12] \\
x_{12}[2] &= [0 \ 0 \ \tau_2 + \tau_3 \ \tau_1]^T u[12] \\
x_{12}[3] &= [0 \ \tau_3 \ \tau_2 \ \tau_1 \ 0]^T u[12] \\
x_{12}[4] &= [\tau_3 \ \tau_2 \ \tau_1 \ 0 \ 0]^T u[12] \\
x_{12}[5] &= [\tau_2 \ 0 \ \tau_1 \ 0 \ 0]^T u[12] \\
x_{12}[6] &= [0 \ \tau_1 \ 0 \ 0 \ 0]^T u[12] \\
x_{12}[7] &= [\tau_1 \ 0 \ 0 \ 0 \ 0]^T u[12] \\
x_{12}[8] &= [\tau_1 \ 0 \ 0 \ 0 \ 0]^T u[12] 
\end{align*}
\]

These vectors, which are specific for the 5\textsuperscript{th} order case with \( \alpha = 1 \), can be used to project ahead the estimates of the observer states, and then equation 4.9 can be used, given an initial guess at what the \( \hat{A} \) is, to determine an A matrix. Given that the \( \tau \)'s of the above column vectors are sample dependent, the time varying matrices can be developed. The following definitions are for a sample interval for the 5\textsuperscript{th} order case with \( \alpha = 1 \).
\[ \gamma[0] = [0 \ 0 \ 0 \ 0 \ 1]^T \] (5.7)
\[ \gamma[1] = [0 \ 0 \ 0 \ \tau 2[p] + \tau 3[p] \ \tau l[p]]^T \] (5.8)
\[ \gamma[2] = [0 \ \tau 3[p] \ \tau 2[p] \ \tau l[p] \ 0]^T \] (5.9)
\[ \gamma[3] = [\tau 3[p] \ \tau 2[p] \ \tau l[p] \ 0 \ 0]^T \] (5.10)
\[ \gamma[4] = [\tau 2[p] \ 0 \ \tau l[p] \ 0 \ 0]^T \] (5.11)
\[ \gamma[5] = [0 \ \tau l[p] \ 0 \ 0 \ 0]^T \] (5.12)
\[ \gamma[6] = [\tau l[p] \ 0 \ 0 \ 0 \ 0]^T \] (5.13)
\[ \gamma[7] = [\tau l[p] \ 0 \ 0 \ 0 \ 0]^T \] (5.14)

where \( p \) relates the above equations to input sample \( u[p] \). The closed form solution for the state update equation can be found from inspection of the iterated values for \( x \). Given some initial value for \( x[0] \), where \( \gamma \) is a function of \( \tau \) which is a function of \( u[k-7-m] \).

\[ x[1] = \gamma[0]u[0] \]
\[ x[2] = \gamma[1]u[0] + \gamma[0]u[1] \]
\[ \vdots \]
\[ x[k+1] = \sum_{m=0}^{7} \gamma[7-m]u[k-7+m] \] (5.15)

These state projections can now be used as the system states of equation 4.9 to determine \( \Delta A \), and thereby attain a plant for the 5th order observer. Similar transition vectors would be developed for systems having different numbers of states and different \( \alpha \)'s. The output of the system can be projected in a similar manner as the state vector projections. Equations 5.16 through 6.23 are generated by observing when equations 5.7 - 5.14 produce an output.
\[ \lambda[0] = 0 \quad (16) \]
\[ \lambda[1] = 0 \quad (17) \]
\[ \lambda[2] = 0 \quad (18) \]
\[ \lambda[3] = \tau_3 \quad (19) \]
\[ \lambda[4] = \tau_2 \quad (20) \]
\[ \lambda[5] = 0 \quad (21) \]
\[ \lambda[6] = 0 \quad (22) \]
\[ \lambda[7] = \tau_1 \quad (23) \]

Equations 5.16 - 5.23 are incorporated into equation 5.24, which is a closed form solution for the output sequence, \( y[k] \).

\[ y[k + 1] = \sum_{m=0}^{7} \lambda[7-m] u[k-7+m] \quad (5.24) \]

Example 5.3 shows how equation 5.7 - 5.24 and equation 4.9 are used to determine the "best" possible observer system based on knowledge of velocity distribution. The velocity distribution of example 5.2 is used in example 5.3.
Example 5.3:

This example uses the velocity distribution of example 5.2 for all the input samples, therefore \( \tau_1=\tau_3=.1 \) and \( \tau_3=.8 \) for all samples. The results of equations 5.7 - 5.14 and 5.16 - 5.23 are given below, where the counting index runs from left to right. The input sequence is shown in figure 5.2.

\[
\begin{pmatrix}
0 & 0 & 0 & 0.1 & 0.8 & 0 & 0.1 & 0.1 \\
0 & 0 & 0.1 & 0.8 & 0 & 0.1 & 0 & 0 \\
0 & 0 & 0.8 & 0 & 0.1 & 0 & 0 & 0 \\
0 & 0.9 & 0.1 & 0.1 & 0 & 0 & 0 & 0 \\
1 & 0.1 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

\( \gamma \)

\[
\lambda^T = (0 \ 0 \ 0 \ 0.1 \ 0.8 \ 0 \ 0 \ 0.1)
\]

\( k := 0..40 \)

**Figure 5.2 - Input Sequence**

**Figure 5.3 - 40 State Projections based on the Input Velocity Distribution**
Figure 5.3 shows the resulting state projections using equation 5.15. The observer system definitions are given below. The "best guess" at the system behavior is that of a perfect delay, therefore, the plant matrix has ones on the upper diagonal. The initial observer state values are zero.

\[
M = \begin{bmatrix}
-2 & -2 & -2 & -2 & -2 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad A_{\text{hat}} = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad B_{\text{hat}} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}, \quad C_{\text{hat}} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}, \quad x_{\text{hat}}^{(0)} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Next, the observer update equations are used to calculate the difference between the system and the observer states so that an identification can be performed.

\[
k = 0, 120
\]

\[
x_{\text{hat}}^{(k+1)} = A_{\text{hat}}x_{\text{hat}}^{(k)} + B_{\text{hat}}u_k - M(y_k - C_{\text{hat}}x_{\text{hat}}^{(k)})
\]

\[
y_{\text{hat}}^{(k)} = C_{\text{hat}}x_{\text{hat}}^{(k)}
\]

\[
k = 0, 20
\]

![Figure 5.4 - The difference between the System and Observer States](image-url)
Equation 4.9 is used to calculate the difference between the system and observer plants. Forty samples are used for the calculation.

\[
k := 0.39
\]

\[
X_{sys}^{(k)} = x^{(k)}
\]

\[
X_{obs}^{(k)} = \hat{x}^{(k)}
\]

\[
X_{sys}^{(1)} := x^{(k+1)}
\]

\[
X_{obs}^{(1)} := \hat{x}^{(k+1)}
\]

\[
\Delta A = \left( X_{sys} X_{sys}^T \right)^{-1} \cdot X_{sys} \cdot \left( (X_{sys} - X_{obs}) - (A_{hat} + M \cdot Chat) \cdot (X_{sys} - X_{obs}) \right)^T
\]

The results of the calculation are:

\[
\Delta A^T =
\begin{bmatrix}
0.152 & -0.037 & -0.016 & 0.004 & -0.003 \\
0.028 & -0.02 & -0.015 & 0.112 & -5.227 \cdot 10^{-4} \\
0.034 & 0.007 & -0.115 & -0.111 & -4.198 \cdot 10^{-4} \\
0.014 & -0.007 & 0.12 & 0.012 & -0.101 \\
0.025 & -0.005 & -0.001 & -0.01 & 0.099
\end{bmatrix}
\]

The entries of the delta matrix are added to the original observer plant and output matrices, as discussed in example 4.1.
The identified plant and output matrices are:

\[
Ahat = \begin{bmatrix}
0.152 & 0.963 & 0.007 & 0 & 0 \\
0 & 0.019 & 0.985 & 0.055 & 0 \\
0 & 0 & 0.007 & 0.889 & 0 \\
0 & 0 & 0 & 0.055 & 0.899 \\
0 & 0 & 0 & 0 & 0.099 \\
\end{bmatrix}
\]

\[
Chat = (0.848, 0.019, 0, 0, 0)
\]

These matrices are used to recalculate the observer states and the observer output without the use of error feedback. Figure 5.5 shows the difference between the system and the new observer states.

\[
k = 0..80
\]

\[
xhat^{k+1} = Ahat \cdot xhat^k + Bhat \cdot u_k
\]

\[
yhat^k = Chat \cdot xhat^k
\]

\[
k = 0..20
\]

Figure 5.5 - The difference between the System and Observer States After the Identification
This example performed an identification of the system, in this case the system is a projection of what the system states are based on the input velocity distribution. Figure 5.5 shows that there is still some error between the states of the observer system and the actual state values obtained from projection, but the error is small, less than 10% of the total state values. This error is due to the transitory nature of the system. The projection scheme requires that sometimes vehicles transition each sample instant, and sometimes they don't. overlaying the vehicle transitions from different input samples results in a non-stationary system, that is, the plant changes with every sample period. Therefore, the best that the identification calculation can do is to pick the matrix which results in the least error between the system and observer states over a given range of samples.

Now the observer is used in conjunction with the actual highway system. The observer is the best that we can make it based on the available information, therefore the error between the system and observer states will be at a minimum. Use of this technique requires that some error is acceptable for the state estimates. The observer will still indicate when a change has occurred in the system through an error at the outputs which is greater than the excepted error.
CHAPTER VI

Non-Linearity of the Highway System

When considering highway traffic it is readily evident that traffic flow is non-linear in nature. When the traffic density picks up and the spacing between vehicles decreases the traffic tends to slow down. If the vehicles are bumper to bumper then chances are they will not be traveling at 60 MPH. A system is said to be non-linear if the states are a function of themselves, not simply a linear combination of the states. There is non-linearity because individual vehicle velocities are a function of the states, the number of vehicles within a sub-control volume. The traffic flow is not only non-linear but is also time-varying as discussed in chapter 5, but also because not all drivers maintain a constant velocity, faster drivers are often forced to slow down when they encounter slower traffic, and steep grades and sharp curves cause vehicles to change velocity. A time varying system has already been shown to be feasible with the highway model. The vast majority of research is in linear systems, not in non-linear systems. However, many time varying linear systems are successfully used to model non-linear systems if some assumptions can be made and adhered to.

This investigation has used a linear, discrete-time, state-variable model to represent the traffic flow. The assumption that is necessary to allow this model to accurately describe non-linear traffic flow is that the sampling is fast enough to eliminate any non-linearity in the model during a sample instant, as discussed in chapter 3, and that the model can be time varying. Non-linearity is forced to occur from one sample period to the next, thereby forcing the linear model to be time varying but constant during each sample period. This technique has been successfully used with linear, state-space systems to mimic non-linear systems. [2]
The mean velocity can be plotted as a function of the vehicle density. Figure 1 shows a typical shape for this curve, but the units will have to be determined from sampling traffic flow. The shape is that of a low pass filter. The vehicle density is a description of how many vehicles there are within a specific length of highway. Obviously, the number of lanes on the highway has an effect on the vehicle density.

![Velocity vs. Vehicles](image)

**Figure 6.1 - A Typical Curve Showing Non-Linear Behavior**

Up to this point, all the models have assumed a linear model for the traffic flow and unlimited capacity for each sub-segment. These two factors are now considered as they relate to the developed model. For each sub-segment of the control volume, the length and width (the number of lanes of traffic) is known. There is some maximum number of vehicles which can fit into each sub-control volume. Once this maximum is reached, the transition of vehicles into this sub-control volume must be corrected so that the state does not exceed this maximum. In this way, the model will mimic traffic behavior as a backup occurs and the length of the slow down increases. Before the state reaches a maximum, the non-linearity will start to effect the model as the mean velocity of vehicles in this state is forced to decrease.
In order to force the non-linear behavior into a time varying, linear behavior, an identification must be performed at each sample instant with a very short sampling time. The identification allows the non-linearity to have an effect on the system, which is then identified, and the effect is seen as a change in the state-space quadruple for the next sample instant.

Non-linear systems are not simulated in this paper. The fact that the model can be used with such systems has been discussed so that future study in this area can build on the model developed here. Time varying systems are considered in chapter 8 where an ARMA canonical model is used to perform a system identification. If some identification of the system can be performed, non-linearities in the system can be modeled.
This chapter is devoted to the investigation of one type of traffic problem that may be encountered, and the observer reaction to this problem. The problem under investigation is a single fault, that is, a single incident which has dramatically effected the ability of vehicles to leave one of the sub-control volumes.

The control volume used in this investigation is five miles in length, and has five, one mile sub-control volumes. It is assumed that the mean vehicle speed is 60 MPH, and the variance of the velocity distributions is small so that all the vehicles can be treated as if they are traveling at the mean velocity. The effect of sampling rate on convergence behavior of the observer is studied.

Each of the following examples will use a different sampling rate. The observer is turned on, with zero initial conditions, at some time after the system has some level of vehicles in the control volume. The observer is initially matched to the system, which would be the case for a "good" model of a well behaved interval of data. After eight minutes the observer states have converged to the system. At this point, an incident occurs in the second sub-control volume, state x4. The observer reaction to this incident is recorded in five plots showing one state each of the system and observer.

Another measurement that is available is that of control volume occupancy. If the monitoring of a control volume can be initiated during a period in which very few vehicles are traveling through it, say late at night, then an accurate occupancy can be obtained by counting the vehicles which enter and leave the control volume. Comparison of this count
to the total number of vehicles the observer has calculated to be in the control volume provides a flag which can be used to decide if an error detected at the output is due to fluctuations in vehicle speeds, or if it is due to some incident within the control volume. The differences between these two occupancy counts will also be examined through the following examples. The final plot in the examples shows all five of the observer states plotted over time. These plots show how fluctuations may spread through each state of the control volumes.
Example 7.1:

All vehicles are travelling at 60 MPH, and the sampling interval is $T = 6$ seconds.

![Graph](image)

**Figure 7.1 - The Input Sequence**

The system and observer are initially matched.

**PLANT MATRICES:**

$$A_{sys} = \begin{bmatrix} .9 & .1 & 0 & 0 & 0 \\ 0 & .9 & .1 & 0 & 0 \\ 0 & 0 & .9 & .1 & 0 \\ 0 & 0 & 0 & .9 & 0 \\ 0 & 0 & 0 & 0 & .9 \end{bmatrix}$$

$$A_{obs} = \begin{bmatrix} .9 & .1 & 0 & 0 & 0 \\ 0 & .9 & .1 & 0 & 0 \\ 0 & 0 & .9 & .1 & 0 \\ 0 & 0 & 0 & .9 & 1 \\ 0 & 0 & 0 & 0 & .9 \end{bmatrix}$$

**INPUT MATRICES:**

$$B_{sys} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$B_{obs} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

**OUTPUT MATRICES:**

$$C_{sys} = (.1 \ 0 \ 0 \ 0 \ 0)$$

$$C_{obs} = (.1 \ 0 \ 0 \ 0 \ 0)$$

**STATE INITIAL CONDITIONS:**

$$X_{sys}^{<0>} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix}$$

$$X_{obs}^{<0>} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
ERROR FEEDBACK MATRIX AND EIGENVALUES:

\[
\begin{bmatrix}
-2 \\
-2 \\
-2 \\
-2
\end{bmatrix}
\]

\[
M = \begin{bmatrix}
-2 \\
-2 \\
-2 \\
-2
\end{bmatrix}
\]

eigenvals(A_{obs} + M \cdot C_{obs}) =

\[
\begin{bmatrix}
0.943 - 0.094i \\
0.943 + 0.094i \\
0.761 \\
0.826 - 0.09i \\
0.826 + 0.09i
\end{bmatrix}
\]

CALCULATION OF SIMULATION RESULTS:

\( k = 0..79 \) (The First 8 Minutes)

\[
X_{sys}^{<k+1>} = A_{sys} \cdot X_{sys}^{<k>} + B_{sys} \cdot U_k
\]

\[
OUTPUT^{<k>} = C_{sys} \cdot X_{sys}^{<k>}
\]

\[
X_{obs}^{<k+1>} = A_{obs} \cdot X_{obs}^{<k>} + B_{obs} \cdot U_k - M \cdot (C_{sys} \cdot X_{sys}^{<k>} - C_{obs} \cdot X_{obs}^{<k>})
\]

NOW THERE IS A PROBLEM IN ONE OF THE SYSTEM STATES, STATE \( x_4 \):

\[
\begin{bmatrix}
0.9 & 0 & 0 & 0
\end{bmatrix}
\]

\[
0.9 \ 1 \ 0 \ 0 \ 0
\]

\[
0 \ 0.9 \ 1 \ 0 \ 0
\]

\[
0 \ 0 \ 0 \ 0.96 \ 0.1
\]

\[
0 \ 0 \ 0 \ 0.9
\]

\[
M = \begin{bmatrix}
-2 \\
-2 \\
-2 \\
-2
\end{bmatrix}
\]

eigenvals(A_{obs} + M \cdot C_{obs}) =

\[
\begin{bmatrix}
0.943 - 0.094i \\
0.943 + 0.094i \\
0.761 \\
0.826 - 0.09i \\
0.826 + 0.09i
\end{bmatrix}
\]

\( k = 80..399 \) (Simulation results for minutes 8 to 40)

\[
X_{sys}^{<k+1>} = A_{sys} \cdot X_{sys}^{<k>} + B_{sys} \cdot U_k
\]

\[
OUTPUT^{<k>} = C_{sys} \cdot X_{sys}^{<k>}
\]

\[
X_{obs}^{<k+1>} = A_{obs} \cdot X_{obs}^{<k>} + B_{obs} \cdot U_k - M \cdot (C_{sys} \cdot X_{sys}^{<k>} - C_{obs} \cdot X_{obs}^{<k>})
\]

\( k = 0..300 \)

\[
Sysum^{<k>} = (1 \ 1 \ 1 \ 1) \cdot X_{sys}^{<k>}
\]

\[
Obsum^{<k>} = (1 \ 1 \ 1 \ 1) \cdot X_{obs}^{<k>}
\]

Calculation of Control Volume Occupancy
Figure 7.2 - System and Observer State Values for the First 30 Minutes
Figure 7.3 - The System and Observer Occupancy Levels For the First 30 Minutes

Figure 7.4 - All Five Observer States for the First 30 Minutes
Example 7.2:

All vehicles are travelling at 60 MPH, and the sampling interval is $T = 9$ seconds.

The input sequence is the that of example 7.1. The calculations are performed in the same manner as in example 7.1. Eight minutes is equivalent to 53 sample periods, and forty minutes is equivalent to 267 sample periods. The differences in the matrices from example 7.1 are:

**PLANT MATRICES:**

\[
\text{Asys} = \begin{bmatrix}
0.7 & 0.3 & 0 & 0 \\
0 & 0.7 & 0.3 & 0 \\
0 & 0 & 0.7 & 0.3 \\
0 & 0 & 0 & 0.7
\end{bmatrix},
\text{Aobs} = \begin{bmatrix}
0.7 & 0 & 0 & 0 \\
0 & 0.7 & 0 & 0 \\
0 & 0 & 0.7 & 0 \\
0 & 0 & 0 & 0.7
\end{bmatrix}
\]

**OUTPUT MATRICES:**

\[
\text{Csys} = (0.3 \ 0 \ 0 \ 0), \quad \text{Cobs} = (0.3 \ 0 \ 0 \ 0)
\]

**ERROR FEEDBACK MATRIX AND EIGENVALUES:**

\[
M = \begin{bmatrix}
-0.667 \\
-0.667 \\
-0.667 \\
-0.667
\end{bmatrix}, \quad \text{eigenvals(Aobs + M.Cobs)} = \begin{bmatrix}
0.858 - 0.24i \\
0.858 + 0.24i \\
0.575 - 0.241i \\
0.575 + 0.241i \\
0.434
\end{bmatrix}
\]

**AFTER 8 MINUTES THERE IS A PROBLEM IN ONE OF THE SYSTEM STATES, STATE x4:**
The system changes to:

\[
\begin{bmatrix}
0.7 & 0.3 & 0 & 0 \\
0 & 0.7 & 0.3 & 0 \\
0 & 0 & 0.8 & 0.3 \\
0 & 0 & 0 & 0.7
\end{bmatrix}
\]

The results of 40 minutes of simulation are shown in figures 7.5 - 7.7.

Figure 7.5 a

Figure 7.5 b

Figure 7.5 c

Figure 7.5 d
Figure 7.5 - System and Observer State Values for the First 30 Minutes

Figure 7.6 - The System and Observer Occupancy Levels For the First 30 Minutes
Figure 7.7 - All Five Observer States for the First 30 Minutes
**Example 7.3:**

All vehicles are travelling at 60 MPH, and the sampling interval is $T = 30$ seconds.

The input sequence is the that of example 7.1. The calculations are the same as example 7.2. Eight minutes equals 16 samples, and forty minutes equals 80 samples.

The initial System and Observer Matrices, and the $M$ matrix differ from example 7.1:

$$
A = \begin{bmatrix}
0.5 & 0.5 & 0 & 0 & 0 \\
0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0.5 & 0.5 & 0 \\
0 & 0 & 0 & 0.5 & 0.5
\end{bmatrix}
\quad C = \begin{bmatrix}
0.5 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
M = \begin{bmatrix}
-0.4 & \text{eigenvals}(A_{obs} + M_{obs}) =
\begin{bmatrix}
0.768 - 0.356i & 0.768 + 0.356i \\
0.327 - 0.363i & 0.327 + 0.363i \\
0.11 & \end{bmatrix}
\end{bmatrix}
$$

After 8 minutes, the system changes to:

$$
A_{sys} = \begin{bmatrix}
0.5 & 0.5 & 0 & 0 & 0 \\
0 & 0.5 & 0.5 & 0 & 0 \\
0 & 0 & 0.5 & 0.2 & 0 \\
0 & 0 & 0 & 0.8 & 0.5 \\
0 & 0 & 0 & 0 & 0.5
\end{bmatrix}
$$

The results of 40 minutes of simulation are shown in figures 7.8 - 7.11.
Figure 7.8 - System and Observer State Values for the First 30 Minutes
Figure 7.9 - The System and Observer Occupancy Levels For the First 30 Minutes

Figure 7.10 - All Five Observer States for the First 30 Minutes
The conclusion based on these three examples is that a fast sampling rate allows all the observer states to converge to the system states, except that state in which the incident occurred. Example 7.3 shows that a 30 second sampling rate did not allow the unaffected states to re-converge, but examples 7.1 and 7.2 show that re-convergence is achieved for the unaffected states when a fast sampling rate is used. One explanation for this may be that for a sampling rate of six seconds, very few vehicles are leaving the state during a sample period. Once the incident occurs and fewer vehicles are leaving the effected state, the change in plant entries is forced to be small, and allows convergence. In order to investigate if convergence is only hindered in the effected state, the next set of examples places a fault in one of the other five states. Sampling is at the fast, 6 second, rate.
Example 7.4:

This example exactly follows example 7.1, until the fault occurs at sample time $k = 80$.

NOW THERE IS A PROBLEM IN ONE OF THE SYSTEM STATES, STATE $x_5$:

\[
A_{sys} = \begin{bmatrix}
0.9 & 0.1 & 0 & 0 & 0 \\
0 & 0.9 & 0.1 & 0 & 0 \\
0 & 0 & 0.9 & 0.1 & 0 \\
0 & 0 & 0 & 0.9 & 0.04 \\
0 & 0 & 0 & 0 & 0.96
\end{bmatrix}
\]

**Figure 7.11 a**

**Figure 7.11 b**

**Figure 7.11 c**

**Figure 7.11 d**
Figure 7.11 - System and Observer State Values for the First 30 Minutes

Figure 7.12 - All Five Observer States for the First 30 Minutes
Example 7.5:

This example exactly follows example 7.1, until the fault occurs.

NOW THERE IS A PROBLEM IN ONE OF THE SYSTEM STATES, STATE $x_2$:

\[
A_{sys} = \begin{bmatrix}
0.9 & 0.03 & 0 & 0 & 0 \\
0 & 0.97 & 0.1 & 0 & 0 \\
0 & 0 & 0.9 & 0.1 & 0 \\
0 & 0 & 0 & 0.9 & 0.1 \\
0 & 0 & 0 & 0 & 0.9
\end{bmatrix}
\]

\[\begin{align*}
\text{Vehicles} & \quad 400 \\
\text{Sample, k} & \quad 0 \quad 100 \quad 200 \quad 300
\end{align*}\]

$- x_1$

$- x_{obs1}$

Figure 7.13 a

\[\begin{align*}
\text{Vehicles} & \quad 1000 \\
\text{Sample, k} & \quad 0 \quad 100 \quad 200 \quad 300
\end{align*}\]

$- x_2$

$- x_{obs2}$

Figure 7.13 b

\[\begin{align*}
\text{Vehicles} & \quad 400 \\
\text{Sample, k} & \quad 0 \quad 100 \quad 200 \quad 300
\end{align*}\]

$- x_3$

$- x_{obs3}$

Figure 7.13 c

\[\begin{align*}
\text{Vehicles} & \quad 400 \\
\text{Sample, k} & \quad 0 \quad 100 \quad 200 \quad 300
\end{align*}\]

$- x_4$

$- x_{obs4}$

Figure 7.13 d
Figure 7.13 - System and Observer State Values for the First 30 Minutes

Figure 7.14 - All Five Observer States for the First 30 Minutes
As in the first set of examples, the state in which the fault occurs does not reconverge to the system state, but all the other states do. Based on the results of these examples, the sampling rate used to collect data values should be relatively fast in order to have re-convergence of the unaffected states after the fault has occurred. If the assumption is that only single faults will occur, that is there is only one incident at a time which will dramatically effect traffic flow, then one observer state will be incorrect after some time, and there may be a way of deciding which sub-control volume has had the incident. Once an error is detected at the output, it can be inferred that something has happened within the system. The sampling rate provides knowledge of how long ago the incident would have occurred for it to have happened in each of the sub-control volumes.

A proposal for further study is to investigate how the single fault assumption might lead to discovery of which state the fault occurred. One investigation might be to implement five test observers, each having the incident occur in one of the five sub-control volumes at the time deduced using the sampling rate. If one of these observers produces a "match" to the system output, then it may indicate that the incident has occurred in this state. In order for the observer output to exactly match the system output using the test observers method just described, the change in observer parameters would have to be exactly the same as that in the system. However, by making adjustments of equal magnitude to one column in each of the five test observes plant matrices, the correct observer may be indicated by the smallest error between the outputs.

The control-volume occupancies are also plotted for the first three example in this chapter. These plots show that once the incident occurs, the occupancy counts diverge. This knowledge provides useful information about when something has changed in the system. If the observer occupancy level is higher than that of the system, it can be inferred that some of the vehicles are traveling faster than when they crossed the input sensor and
there must not be any impedance's in the system, so there are no problems. If the observer occupancy is less than the system occupancy, then the vehicles are not leaving the system when they should have, based on the velocity distributions of the input. Another proposal for further study would be to investigate how this discrepancy in vehicle counts can be used to correct the observer states. Five test observers can be used, as described above, but now the error in occupancies is fed back to one of the five states. If this correction in the state value provides a match to the measured system output, then this would indicate the effected state.

The control volume occupancy is a unique artifact of the model. Very few systems provide a linear combination of all the states as a measurable value. Further study into how this value can be used in an identification process needs to be pursued.

The final plot of each example shows all five states over time. A categorization of how each state reacts to the single fault may lead to being able to recognize which state the fault occurred by observation of the states following the fault. If the behavior of the states is similar during re-convergence for these states, and it is different than the pattern of the states located before the fault, this plot may lead to identifying the location of the fault. This study is suggested for future research.
CHAPTER VIII

System Identification Using

An ARMA Model

Chapter 4 introduced one method of identifying the system based on knowledge of the system states. This chapter proposes another modeling technique which does not require knowledge of the states as they have been defined for the highway traffic model. The ARMA (Auto Regressive Moving Average) model is a canonical form of the state-space equations. The special feature of this model is that the states of the system are past input and output values of the system, which, by definition of the highway model, are measured. The state-space quadruple of the ARMA model can be identified using a least squares calculation involving only the output equation. By mapping the state-space quadruple of the highway system model into the ARMA canonical form, performing the identification of the ARMA model, and mapping these matrix elements back to the system model, the system can be identified. This chapter presents the state-space quadruple mapping between the two system models and shows two examples which perform highway system identification through ARMA system identification.

The matrix elements of the observer canonical form, the control canonical form, and the ARMA canonical form of state-space equations are closely related. [4] The coefficients of a difference equation in the z domain can be mapped directly into these forms. Given a third order difference equation in a general form,

$$ G(z) = \frac{b_1 z^2 + b_2 z + b_3}{z^3 + a_1 z^2 + a_2 z + a_3}, \quad (8.1) $$
the parameters can be collected into a vector,
\[
\theta = [-a_1 \ -a_2 \ -a_3 \ b_1 \ b_2 \ b_3]^T.
\]  (8.2)

Mapping of the elements of the \( \theta \) vector of equation 8.2 directly into the observer, control, and ARMA canonical forms is shown in equations 8.3, 8.4 and 8.5.

\[
A_O = \begin{bmatrix} 0 & 0 & -a_3 \\ 1 & 0 & -a_2 \\ 0 & 1 & -a_1 \end{bmatrix} \quad B_O = \begin{bmatrix} b_3 \\ b_2 \\ b_1 \end{bmatrix} \quad C_O = [0 \ 0 \ 1] \quad (8.3)
\]

\[
A_C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \quad B_C = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C_C = [b_3 \ b_2 \ b_1] \quad (8.4)
\]

\[
A_A = \begin{bmatrix} -a_1 & -a_2 & -a_3 & b_1 & b_2 & b_3 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \quad B_A = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (8.5)
\]

\[
C_A = [-a_1 \ -a_2 \ -a_3 \ b_1 \ b_2 \ b_3]
\]

All three of these state-space models use the general form of the recursive update equations of equation 8.6.

\[
\begin{align*}
x[k+1] &= A x[k] + B u[k] \\
y[k] &= C x[k]
\end{align*}
\]  (8.6)

However, the states of each of the three models are different. A linear transformation can be performed on any state-space quadruple to transform it into another state-space quadruple, but the meaning of the states will be changed. [1] A linear transformation of the highway state-space model into any of the above canonical forms is possible, but the state definition will be changed from that of the number of vehicles in a sub-control
volume. An inverse transformation can be performed to re-acquire the original state definition.

Transformation of a state-space quadruple into the control canonical form is performed by using a specific transformation matrix, Q. This transformation is used to obtain the elements of equation 8.2 which can then be directly inserted into the ARMA model, equation 8.5. Calculation of the transformation matrix Q is described below.

Generation of the Q transformation matrix

Given any $n^{th}$ order state-space system model having matrices A, B, and C, the controllability matrix is used to generate the rows of the Q matrix. The controllability matrix is defined as the $S$ matrix, where

$$S = [s_1 \ s_2 \ s_3 \ \ldots \ s_{2n}] \quad (8.7)$$

and

$$s_1 = B$$
$$s_2 = As_1$$
$$s_3 = As_2$$
$$\vdots$$
$$s_N = As_{2n}$$

If $S$ is of rank $2n$, then the system is controllable. The first row of the Q matrix is given in equation 8.8, which leads to equation 8.9. The Control Canonical state-space matrices are acquired through use of equation 8.10.
The mapping of a general 2\textsuperscript{nd} order state-space system into the control canonical form is shown next. The elements of the A, B, and C matrices are left as variables.
Mapping of a general 2\textsuperscript{nd} order state-space model into the control canonical form.

The general form of the system is:

\[
A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \quad B = \begin{pmatrix} a_5 \\ a_6 \end{pmatrix} \quad C = \begin{pmatrix} a_7 & a_8 \end{pmatrix}
\]

Calculation of the controllability matrix using equation 8.7:

\[
S_1 = B = \begin{pmatrix} a_5 \\ a_6 \end{pmatrix}
\]

\[
S_2 = A*S_1 = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \begin{pmatrix} a_5 \\ a_6 \end{pmatrix} = \begin{pmatrix} a_1a_5 + a_3a_6 \\ a_2a_5 + a_4a_6 \end{pmatrix}
\]

\[
S_2 = \begin{pmatrix} a_1a_5 + a_3a_6 \\ a_2a_5 + a_4a_6 \end{pmatrix}
\]

\[
S = [ S_1 \mid S_2 ] = \begin{pmatrix} a_5 & a_1a_5 + a_3a_6 \\ a_6 & a_2a_5 + a_4a_6 \end{pmatrix}
\]

Calculation of the transformation matrix, Q, using equations 8.8 & 8.9:

\[
S^{-1} = \begin{pmatrix} (a_2a_5 + a_4a_6) & -(a_1a_5 + a_3a_6) \\ (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) & (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) \\ -a_6 & a_5 \\ (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) & (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) \end{pmatrix}
\]

\[
Q_1^T = (0 \ 1) S^{-1}
\]

\[
Q_1^T = (0 \ 1) \begin{pmatrix} (a_2a_5 + a_4a_6) & -(a_1a_5 + a_3a_6) \\ (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) & (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) \\ -a_6 & a_5 \\ (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) & (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) \end{pmatrix}
\]

\[
Q_1^T = \begin{pmatrix} (a_2a_5 + a_4a_6) & -(a_1a_5 + a_3a_6) \\ (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) & (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) \\ -a_6 & a_5 \\ (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) & (a_2a_5^2 + a_5a_4a_6 - a_6a_1a_5 - a_3a_6^2) \end{pmatrix}
\]
Calculation of the Control Canonical matrices using equation 8.10:

\[ A_c = QAQ^{-1} \]

\[ A_c = \begin{pmatrix} 0 & 1 \\ a1 \cdot a4 + a3 \cdot a2 & a4 + a1 \end{pmatrix} \quad (8.11) \]

\[ B_c = QC \]

\[ B_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (8.12) \]

\[ C_c = CQ^{-1} \]

\[ C_c = (a7 \cdot a3 \cdot a6 - a7 \cdot a5 \cdot a4 - a8 \cdot a6 \cdot a1 + a8 \cdot a2 \cdot a5 \cdot a7 \cdot a5 + a8 \cdot a6) \quad (8.13) \]

Equation 8.4, with a change of variable, is:

\[ A_c = \begin{pmatrix} 0 & 1 \\ -b4 & -b3 \end{pmatrix} \quad B_c = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad C_c = (b2 \ b1) \quad (8.14) \]
Equations 8.11, 8.12 and 8.13 provide the mapping from any general second order state-space system into the control canonical form. Equating terms between equations 8.11 - 8.14, there are four equations and eight unknowns, the \( b \) terms are the known variables obtained from an identification of the ARMA model. The four equations are shown in equation 8.15

\[
\begin{align*}
\begin{bmatrix}
    b_1 &= a_7 a_5 + a_8 a_6 \\
    b_2 &= a_7 a_3 a_6 - a_7 a_5 a_4 - a_8 a_6 a_1 + a_8 a_2 a_5 \\
    b_3 &= a_1 a_4 - a_3 a_2 \\
    b_4 &= -a_4 - a_1
\end{bmatrix}
\end{align*}
\] (8.15)

The \( b \) variables of equation 8.15 are exactly the variable entries of the ARMA model in equation 8.5. Now the ARMA model can be used to perform an identification of the \( a \) variables in equation 8.5.

The states of the \( n^{th} \) order highway system and the ARMA system are defined as \( x[k] \) and \( \phi[k] \), respectively, for the following discussion. The inputs and outputs of the two systems are identical, therefore \( y[k] \) is defined as the output for both systems. As discussed earlier, the ARMA system states are defined as

\[
\phi[k] = [-y[k-1] - y[k-2] \cdots - y[k-n] \ u[k-1] \ \cdots \ u[k-n]]^T. 
\] (8.16)

From equations 8.2 and 8.5, it is apparent that \( C_A = 0 \). Therefore, from equation 8.6 the state update and output equations for the ARMA system are

\[
\begin{align*}
\phi[k+1] &= A_A \phi[k] + B_A u[k], \\
y[k] &= C_A \phi[k].
\end{align*}
\] (8.17) (8.18)

Equation 8.18 is made up of inputs and outputs of the highway system model and a parameter vector which linearly relates the ARMA state-space quadruple entries to those of the highway model. The over determined form of equation 8.18, containing \( N \) sets of solutions and \( n \) states, is
\[ Y = \theta \Phi, \quad (8.19) \]

where
\[ Y = Y[k] = [y[k-N] \cdots y[k]], \quad (8.20) \]

and
\[ \Phi = \Phi[k] = [\phi[k-N] \cdots \phi[k-1] \phi[k]]^T. \quad (8.21) \]

The "normal equations" for use with a batch least squares calculation are given in equation 8.16. [4] The derivation to reach this equation is not shown here.

\[ \Phi^T \Phi \hat{\theta}_{LS} = \Phi^T Y \quad (8.22) \]

Solving for the \( \theta \) variable in equation 8.2, the least squares estimate of equation 8.2, results in

\[ \hat{\theta}_{LS} = (\Phi^T \Phi)^{-1} \Phi^T Y. \quad (8.23) \]

Equation 8.23 is the batch, least squares calculation of the ARMA state-space quadruple entries. Mapping \( \hat{\theta}_{LS} \) to the highway model, through use of equation 8.15, identifies the highway system.

The \( a \) variables in equation 8.15 cannot presently be solved for. However, the highway system model has a specific form which eliminates some of the terms in equation 8.15, and can be even further constrained if the sampling rate is fast enough. An \( n^{\text{th}} \) order highway model will have zero entries below the main diagonal of the \( A \) matrix. If sampling is fast enough to prevent any vehicles from transitioning more than one state ahead, then the only non-zero entries in the \( A \) matrix are the main diagonal and the first upper diagonal. The only non-zero entries in the \( B \) and \( C \) matrices are entries \((1,n)\) and \((1,1)\), respectively, that is, vehicles entering the system must enter only the first state, and vehicles leaving the system must come from only the last state. These constraints force \( a2 = a5 = a8 = 0 \) in equation 8.15. Further constraints of the highway model are that the columns of the state-space quadruple must sum to 1, and all entries must be positive and less than or equal to 1. When these special conditions of the highway model are
\[ Y = \theta \Phi, \quad (8.19) \]

where
\[ Y = Y[k] = [y[k-N] \ldots y[k]], \quad (8.20) \]

and
\[ \Phi = \Phi[k] = [\phi[k-N] \ldots \phi[k-1] \phi[k]]^T. \quad (8.21) \]

The "normal equations" for use with a batch least squares calculation are given in equation 8.16. [4] The derivation to reach this equation is not shown here.
\[ \Phi^T \Phi \hat{\theta}_{LS} = \Phi^T Y \quad (8.22) \]

Solving for the \( \theta \) variable in equation 8.2, the least squares estimate of equation 8.2, results in
\[ \hat{\theta}_{LS} = (\Phi^T \Phi)^{-1} \Phi^T Y. \quad (8.23) \]

Equation 8.23 is the batch, least squares calculation of the ARMA state-space quadruple entries. Mapping \( \hat{\theta}_{LS} \) to the highway model, through use of equation 8.15, identifies the highway system.

The \( a \) variables in equation 8.15 cannot presently be solved for. However, the highway system model has a specific form which eliminates some of the terms in equation 8.15, and can be even further constrained if the sampling rate is fast enough. An \( n^{th} \) order highway model will have zero entries below the main diagonal of the \( A \) matrix. If sampling is fast enough to prevent any vehicles from transitioning more than one state ahead, then the only non-zero entries in the \( A \) matrix are the main diagonal and the first upper diagonal. The only non-zero entries in the \( B \) and \( C \) matrices are entries \( (1,n) \) and \( (1,1) \), respectively, that is, vehicles entering the system must enter only the first state, and vehicles leaving the system must come from only the last state. These constraints force \( a_2 = a_5 = a_8 = 0 \) in equation 8.15. Further constraints of the highway model are that the columns of the state-space quadruple must sum to 1, and all entries must be positive and less than or equal to 1. When these special conditions of the highway model are
Example 8.1:

The system under investigation is a 2 mile control volume with 2 sub-control volumes of 1 mile each. The traffic flow has been uneventful for some time. The counting index is reset to 0 and an identification is performed using 15 sets of input/output data.

System Set Up

The System Model is:

\[
A = \begin{bmatrix} 0.7 & 0.18 \\ 0 & 0.82 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0.3 \\ 0 \end{bmatrix} \quad x^0 = \begin{bmatrix} 39 \\ 35 \end{bmatrix}
\]

The input sequence is sinusoidal and random, with a mean of 40 and a variance of 4.

![Vehicles](image.png)

Figure 8.1 - The Input Sequence

Calculation of 20 system states and system outputs.

\[
k = 0 \cdots 20 \\
\begin{align*}
\dot{x}^{k+1} & := A x^k + B u_k \\
y^k & := C x^k
\end{align*}
\]
Identification of System:

15 states of the ARMA model are generated, starting with the sample values at \( k=0 \) for the input and output sequences.

\[
k := 2..17
\]

Overdetermined ARMA states

\[
\begin{align*}
\phi^{\langle k-2 \rangle} &= \begin{bmatrix} y_{0,k-1} \\ y_{0,k-2} \\ u_{k-1} \\ u_{k-2} \end{bmatrix} \\
\theta &= (Y \phi^T)(\phi \phi^T)^{-1}
\end{align*}
\]

Least Squares Calculation

\[
\theta = (1.52, -0.574, 2.842 \times 10^{-14}, 0.054)
\]

Initial Guesses at Solutions

\[
a_1 := .75 \quad a_4 := .75 \quad a_6 := .8 \quad a_3 := .25 \quad a_7 := .25
\]

Given

\[
-a_1 \cdot a_4 = \theta_{0,1} \quad a_1 + a_4 = \theta_{0,0} \quad a_1 \geq 0 \quad a_4 \geq 0 \quad a_3 = 1 - a_4 \quad a_7 = 1 - a_1 \quad a_7 \cdot a_3 \cdot a_6 = \theta_{0,3}
\]

Figure 8.2 - Simulation Results of the First 5 Minutes
\[
\begin{bmatrix}
A_{i,0,0} \\
A_{i,0,1} \\
A_{i,1,1}
\end{bmatrix} = \text{Find}(a_1, a_3, a_4, a_6, a_7)
\]

\[
\begin{bmatrix}
B_{i,0} \\
C_{i,0,0}
\end{bmatrix}
\]

Solutions are mapped into the Identified Matrices

\[
A_{1,0} = 0, \quad B_{i,0} = 0, \quad C_{i,1} = 0
\]

Identified System:

\[
A_i = \begin{pmatrix}
0.7 & 0.18 \\
0 & 0.82
\end{pmatrix}, \quad B_i = \begin{pmatrix}
0 \\
1
\end{pmatrix}, \quad C_i = \begin{pmatrix}
0.3 & 0
\end{pmatrix}
\]

The highway model state-space quadruple is successfully identified.
Now that this technique has proven to provide an identification of the system, this model can be used as the observer model discussed in previous chapters to mimic the system and detect any changes which occur in the system. Also, the observer can be used to perform test simulation to help make traffic control decisions. The next investigation using this method is to try to identify a system while it is encountering a fault, that is, use the technique with a changing system. This identification is best performed using a recursive least squares calculation. [4] A weighting matrix, W, is introduced into equation 8.23 to discount past state values and rely more heavily on the most recent states resulting in

\[
\hat{\theta}_{\text{WLS}} = (\Phi^T W \Phi)^{-1} \Phi^T W Y. \tag{8.25}
\]

The weighting matrix is \((N \times N)\), where \(N\) is the number of states used in the over determined solution set, and is defined as

\[
W = \begin{bmatrix}
  a y^{N-1} & 0 & \cdots & 0 \\
  0 & a y^{N-2} & 0 & \cdots \\
  \vdots & \ddots & \ddots & \ddots \\
  0 & \cdots & 0 & a y \\
 0 & \cdots & 0 & 0 & a
\end{bmatrix}. \tag{8.26}
\]

The entries in the \(W\) matrix are determined by \(a\) and \(\gamma\). If \(a = \gamma = 1\), then the calculation is ordinary least squares and equation 8.25 is the same as equation 8.23. If \(a = 1 - \gamma\), then equation 8.25 is an exponentially weighted least squares calculation. The memory length is defined as the number of past state values which are considered significantly in equation 8.25, and is given by \(\frac{1}{\gamma(1 - \gamma)}\). [4] The recursive, weighted least squares equation is defined as
\begin{equation}
\hat{\Theta}_{RWLS}[k+1] = \hat{\Theta}_{RWLS}[k] + L[k+1] [y[k] - \phi^T[k+1] \hat{\Theta}_{RWLS}[k]], \quad (8.27)
\end{equation}

where
\begin{equation}
L[k+1] = \frac{P[k]}{\gamma} \phi[k+1] \left[ \frac{1}{a} + \frac{\phi^T[k+1] P[k] \phi[k]}{\gamma} \right]^{-1} \quad (8.28)
\end{equation}

and
\begin{equation}
P[k] = \left[ \Phi^T[k] \mathcal{W} \Phi[k] \right]^{-1}. \quad (8.29)
\end{equation}

The evaluation of equation 8.27 is performed through the following steps of the identification algorithm. [4]

1. Determine the desired values for $a$, $\gamma$, and $N$.
2. Create $\Phi[k]$ (from equation 8.21) and $Y[k]$ (from equation 8.20).
3. Find the initial values for $\hat{\Theta}_{RWLS}(k)$, through a batch, weighted least squares calculation (from equation 8.25), and $P[k]$ (from equation 8.29).
4. Calculate the highway model parameters (from equation 8.24)
5. Collect $y[k+1]$ and $u[k+1]$.
6. Calculate $L[k+1]$ (from equation 8.25).
7. Calculate $\hat{\Theta}_{RWLS}[k+1]$ (from equation 8.24).
8. Calculate $P(k+1)$ as
\begin{equation}
P(k+1) = \frac{1}{\gamma} \left[ I - L(k+1) \phi^T(k+1) \right] P(k). \quad (8.30)
\end{equation}
9. Calculate the highway model parameters (from equation 8.24)
10. Increment $k$ and go to step 5.

The weighted, recursive least squares algorithm is utilized in example 8.2. In this example the system is changed at sample time $k = 10$ in order to investigate the ability of the above algorithm to identify a changing highway system.
Example 8.2 - ARMA Recursive Least Squares Identification

**System Set Up**

The system initially has these entries in the state space quadruple.

\[
\begin{align*}
    k &= 0..10 \\
    A_{00,k} &= 0.7 & A_{01,k} &= 0.3 & B_{00,k} &= 0 & C_{00,k} &= 0.3 & C_{01,k} &= 0 \\
    A_{10,k} &= 0 & A_{11,k} &= 0.7 & B_{10,k} &= 1 \\
    k &= 11..50 \\
    A_{00,k} &= 0.9 & A_{01,k} &= 0.15 & B_{00,k} &= 0 & C_{00,k} &= 0.1 & C_{01,k} &= 0 \\
    A_{10,k} &= 0 & A_{11,k} &= 85 & B_{10,k} &= 1
\end{align*}
\]

At this sample time, the system changes to these entries.

The input sequence is sinusoidal and random, with a mean of 40 and a variance of 4.

\[
k = 0..60
\]

**Figure 8.3 - Input Sequence**

\[
k = 0..50
\]

The state update equations are:

\[
\begin{align*}
    x^{<k+1>} &= \begin{pmatrix} A_{00,k} & A_{01,k} \\ A_{10,k} & A_{11,k} \end{pmatrix} x^{<k>} + \begin{pmatrix} B_{00,k} \\ B_{10,k} \end{pmatrix} u_k \\
    y^{<k>} &= \begin{pmatrix} C_{00,k} & C_{01,k} \end{pmatrix} x^{<k>}
\end{align*}
\]
Identification of System:

An identification is performed every 6 samples equation 8.24. The steps of the identification algorithm are followed. The results are discussed at the end of the calculation section.

Step 1:

\[ N=6 \quad \gamma = 0.666 \]

\[ a = 1 - \gamma \]

Setting \( a = 1 - \gamma \) makes the the weighting exponential. The memory length is appr.

\[ \frac{1}{1 - \gamma} = 2.994 \]

\[ W = \begin{bmatrix}
0.044 & 0 & 0 & 0 & 0 & 0 \\
0 & 0.066 & 0 & 0 & 0 & 0 \\
0 & 0 & 0.099 & 0 & 0 & 0 \\
0 & 0 & 0 & 0.148 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.222 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.334
\end{bmatrix} \]
The states of the ARMA system are \( \phi \). The first state is at \( k=2 \) because the
input sequence starts at \( k=0 \) and the ARMA state contains \( u[k-2] \).

\[
k = 2 \ldots 50
\]

\[
\phi^{<k>}: = \begin{bmatrix}
-y_{0,k-1} \\
y_{0,k-2} \\
u_{k-1} \\
u_{k-2}
\end{bmatrix}
\]

Step 2:

\[
k = 0 \ldots 5
\]

\[
\Phi_N^{<k>} = \Phi^{<k+2>} = \Phi(N).
\]

\[
\Phi_N = \Phi_N^T
\]

\[
\begin{bmatrix}
-12.727 & -9.99 & 42.219 & 41.666 \\
-16.673 & -12.727 & 39.591 & 42.219 \\
-20.905 & -16.673 & 37.988 & 39.591 \\
-24.661 & -20.905 & 37.603 & 37.988 \\
-27.701 & -24.661 & 34.126 & 37.603
\end{bmatrix}
\]

\[
\Phi_N = \begin{bmatrix}
12.727 \\
16.673 \\
20.905 \\
24.661 \\
27.701 \\
30.081
\end{bmatrix}
\]

\[
k := 5
\]

Step 3:

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & -1.4 \\
0 & 0 & 0 & 0 & 0 & 0.49 \\
0 & 0 & 0 & 0 & 0 & 1.794 \cdot 10^{-13} \\
0 & 0 & 0 & 0 & 0 & 0.09
\end{bmatrix}
\]

WLS Result
Step 4:

The mapping of the ARMA system coefficients into the Highway system coefficients is performed by solving a set of simultaneous equations.

\[ a_1 = .75 \quad a_4 = .75 \quad a_6 = .8 \quad a_3 = .25 \quad a_7 = .25 \quad \text{Initial Guesses at Solutions} \]

Given

Solving Simultaneous Equations

\[ a_1 \cdot a_4 = 0 \quad -a_1 + -a_4 = 0 \quad a_1 \geq 0 \quad a_4 \geq 0 \quad a_3 \equiv 1 - a_4 \quad a_7 \equiv 1 - a_1 \quad a_7 \cdot a_3 \cdot a_6 = 0 \]

\[ k = 5 \]

\[ \begin{bmatrix} A_{i00}^k \\ A_{i01}^k \\ A_{i11}^k \\ B_{i01}^k \\ C_{i00}^k \end{bmatrix} = \text{Find}(a_1, a_3, a_4, a_6, a_7) \]

Solutions are mapped into the Identified Matrices

\[ A_{i00}^k = 0 \quad B_{i00}^k = 0 \quad C_{i01}^k = 0 \]

The identification results:

\[
\begin{pmatrix}
A_{i00}^k & A_{i01}^k \\
A_{i10}^k & A_{i11}^k \\
B_{i00}^k & B_{i01}^k \\
C_{i00}^k & C_{i01}^k
\end{pmatrix} =
\begin{pmatrix}
0.7 & 0.3 \\
0 & 0.7 \\
0 & 0 \\
0 & 0
\end{pmatrix}
\]

Step 5: The next samples are collected.

\[ m = 0 \ldots 3 \]

Step 6:

\[ L^{<k+1>} = \frac{\gamma}{k} \phi^{<k+1>} \left( \frac{1}{a} + \phi^{<k+1>} \frac{1}{\gamma} PN \phi^{<k+1>} \right)^{-1} \]

Step 7:

\[ \theta^{<k+1>} = \theta^{<k>} + L^{<k+1>} \left( \gamma \phi_{k+1} - \phi^{<k+1>} \phi^{<k+1>} \right) \]

Step 8:

\[ PN^{<m>} = \frac{1}{\gamma} \left( \text{identity}(4) - L^{<k+1>} \phi^{<k+1>} \right) PN^{<m>} \]
Step 9:

Given

\[ a_1 - a_4 = \theta_{1,k_1} \quad a_1 \geq 0 \quad a_4 \geq 0 \quad a_3 = 1 - a_4 \quad a_7 = 1 - a_1 \quad a_7 a_3 a_6 = \theta_{3,k_3} \]

\[
\begin{bmatrix}
A_{i00_k} \\
A_{i01_k} \\
A_{i11_k} \\
B_{i00_k} \\
C_{i00_k}
\end{bmatrix}
\]

\[ k = 5 \]

\[ A_{i10_k} = 0 \quad B_{i00_k} = 0 \quad C_{i01_k} = 0 \]

\[ A_{i11_k} := \text{Find}(a_1, a_3, a_4, a_6, a_7) \]

\[ B_{i10_k} \]

\[ C_{i00_k} \]

Step 10:

\[ m = 0, 3 \]

\[ k = k + 1 \]

\[ L^{k+1} = \frac{\gamma L^{k+1}}{L^{k+1}} \]

\[ \theta^{k+1} = \theta^{k+1} + L^{k+1} \left( y_{0,k+1} \phi^{k+1} \theta^{k+1} \right) \]

\[ P_{N^{m+1}} = \left( \text{identity(4)} - L^{k+1} \phi^{k+1} \theta^{k+1} \right)^T P_{N^{m+1}} \]

Given

\[ a_1 - a_4 = \theta_{1,k_1} \quad a_1 \geq 0 \quad a_4 \geq 0 \quad a_3 = 1 - a_4 \quad a_7 = 1 - a_1 \quad a_7 a_3 a_6 = \theta_{3,k_3} \]

\[
\begin{bmatrix}
A_{i00_k} \\
A_{i01_k} \\
A_{i11_k} \\
B_{i00_k} \\
C_{i00_k}
\end{bmatrix}
\]

\[ k = 6 \]

\[ A_{i10_k} := 0 \quad B_{i00_k} := 0 \quad C_{i01_k} := 0 \]

\[ A_{i11_k} := \text{Find}(a_1, a_3, a_4, a_6, a_7) \]

\[ B_{i10_k} \]

\[ C_{i00_k} \]

The results of many repetitions of the above equations are shown on the next page.
Figure 8.5 - The Identified Plant Entries

Figure 8.6 - The Identified Input and Output Matrix Entries

Figure 8.7 - The difference between the System and the Identified Entries
In figures 8.5 and 8.6 when the entry values go to zero, there was no solution found at this sampling instant. These un-identifiable points occur initially when the system changes, at \( k = 11 \). The error in the identified entries approaches zero, as seen in figure 8.7. The weighting matrix has an effect on how the past values of the \( \theta \) matrix are used in the identification calculation. Decreasing the memory length will weight the most recent values more heavily. The \( W \) matrix is changed to the values shown below resulting in a shorter memory length.

\[
W = \begin{bmatrix}
9 \cdot 10^{-6} & 0 & 0 & 0 & 0 & 0 \\
0 & 9 \cdot 10^{-4} & 0 & 0 & 0 & 0 \\
0 & 0 & 9 \cdot 10^{-4} & 0 & 0 & 0 \\
0 & 0 & 0 & 0.009 & 0 & 0 \\
0 & 0 & 0 & 0 & 0.09 & 0 \\
0 & 0 & 0 & 0 & 0 & 0.9
\end{bmatrix}
\]

The identification results from this new weighting matrix are shown below.

**Figure 8.8 - The difference between the System and the Identified Entries**

*After a change in the weighting matrix.*

This new weighting matrix with a shorter memory results in no errors between the actual system parameter entries and those produced from the identification. Also, the identification produces a steady state value approximately 20 samples faster.
Figures 8.7 and 8.8 show that the weighted, recursive least squares algorithm has come close to correctly identifying the changed system for a memory length of 3, and exactly identifying the changed system for a memory length of 1. The shorter memory length also reaches a steady state value for the identification in fewer samples. Further investigation is necessary to determine if there is an optimal sampling rate and weighting matrix which provides a "best" identification.
CHAPTER IX

Conclusions and Future Studies

A SISO, linear, discrete time state-variable model has been developed which directly relates to the highway surface. Two consecutive vehicle sensors, spaced some distance apart on the highway surface, provide a measurement of the input and output to the system. The distance between two consecutive sensors is the control volume. The distance between sensors is divided into smaller lengths, sub-control volumes, and the number of vehicles in each of these sub-control volumes are the states of the system. The form of the state-space quadruple is highly structured. The entries of the state-space quadruples are the percentages of vehicles of each state which transition to another sub-control volume, or remain in the current one. By sampling at a rate that forces a vehicle traveling at the mean velocity to take more than 1.5 sample periods to traverse a sub-control volume, the non-zero entries of the state-space quadruple are restricted to the main diagonal and the first upper diagonal of the square state-transition matrix, to the last entry of the column input matrix, and to the first entry in the row output matrix.

This model can be easily modified to a MIMO system, for handling on and off ramp traffic, by adding or subtracting these new inputs. Sensors must be placed on all ramps in order to maintain an accurate vehicle count. Consecutive control volumes can be cascaded, in order to cover the whole highway system, by using the output of one model as the input to the proceeding model.

The state-space observer is shown to provide useful information about changing systems. A smaller sampling interval provides better results when trying to identify where a single fault has occurred. Convergence behavior is directly controlled by the observer
eigenvalue position in the $z$ plane. As the entries of the state transition matrix move off the main diagonal, indicating slower sampling because more vehicles will leave a state during a sample period, the eigenvalues move toward the origin, and convergence requires fewer samples. As the entries of the state transition matrix move onto the main diagonal, indicating faster sampling because more vehicles remain in their current state during a sampling period, the eigenvalues move toward the unit circle, and convergence requires more sample periods. However, the overall time to convergence is reduced with a faster sampling rate. In the single fault case, a fast sampling rate allows all the observer states re-converge to the system states after the fault, except that state where the fault has occurred. A proposed area of future study is to classify the behavior of the observer states after a fault has occurred in order to determine if those states downstream from the fault behave differently than those prior to the fault. If they do, then the position of the fault can be determined. Another study involving observer behavior would be to investigate the use of multiple observers after a change in the system has been discovered. Each observer would consider the fault to have occurred in a different sub-control volume, and the best match to system behavior may lead to determination of the fault location.

A unique result of the model is that the control volume occupancy, the total number of vehicles contained in a control volume, is a measurable value. In the model, the control volume occupancy is the sum of all the states. On the road surface, the control volume occupancy can be measured by starting the system during a lull in traffic, and then keeping an accurate count of the number of vehicles which enter and leave the control volume. When an incident occurs within the system, the control volume occupancy of the observer diverges from that of the system. Knowledge of this difference provides additional information for identification of the changing system. Investigation of how this singular property can be used for system identification needs to be pursued in any future studies involving this model.
A technique for incorporation of sample velocity distributions into the development of the observer model has been introduced. A batch least-squares calculation is used to identify the best model for the current set of data. By incorporating the velocity distributions into the observer development, the smallest difference between the observer model and the actual system is generated. This results in a smaller steady state error between the observer and system states, and therefore, provides a better estimate as to the values of the system states.

The relationship between the highway system model state-space quadruple and the ARMA model state-space quadruple is derived. This relationship is used to allow the mapping of an identified ARMA model into the highway model. Identification of the ARMA model using a recursive least-squares calculation is successfully obtained for a transitory system. In order for a linear system to model the nonlinear traffic flow, the assumption is made that a fast sampling rate forces the non-linearity to effect the system between samples. The system is considered to be linear during sampling. The changes in the system caused by the non-linearity are identified, and the model is updated before the next sample. Use of the ARMA identification algorithm, or another identification technique, is crucial for modeling the non-linear, transitory traffic flow.

This paper has presented an initial study into the modeling of traffic flow on an interstate highway. The model has some unique properties that should make it useful if implemented. This study has raised many questions that need to be pursued, as good research should.
APPENDIX A

The Mathcad programs used in examples 3.1-3.3 are shown here. There is a floppy disk included with this document that contains the programs used for all the examples of this paper. These programs can be edited using Mathcad 4.0. Changing the state-space quadruple entries is easily accomplished. The reader is encouraged to use these programs to investigate other traffic situations.
Example 3.1:

Input Sequence:

\[ k = 0 \ldots 14 \]
\[ u_k = 30 \]
\[ k = 15 \ldots 19 \]
\[ u_k = 30 + 4 \cdot (k - 14) \]
\[ k = 20 \ldots 24 \]
\[ u_k = 50 \]
\[ k = 25 \ldots 29 \]
\[ u_k = 50 - 6 \cdot (k - 24) \]
\[ k = 30 \ldots 49 \]
\[ u_k = 20 \]

The PLANT matrix, A, is defined as:

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The INPUT matrix, B, is defined as:

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

The OUTPUT matrix, C, is defined as:

\[
C = (1 \ 0 \ 0 \ 0 \ 0)
\]

The initial conditions:

\[
X_{\text{sys}}^{\leq 0} = \begin{bmatrix}
30 \\
30 \\
30 \\
30 \\
30
\end{bmatrix}
\]
\[
X_{\text{obs}}^{\leq 0} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]

\[ k = 0 \ldots 49 \]
The state update equations are:

\[ k = 0..49 \]

\[
X_{sys}^{<k+1>} := A \cdot X_{sys}^{<k>} + B \cdot u_k
\]

\[
Y_{sys}^{<k>} = C \cdot X_{sys}^{<k>}
\]

\[
X_{obs}^{<k+1>} := A \cdot X_{obs}^{<k>} + B \cdot u_k - M \cdot C \cdot (X_{sys}^{<k>} - X_{obs}^{<k>})
\]
b)

\[ M = \begin{bmatrix}
-0.5 \\
-0.3 \\
-0.1 \\
-0.05 \\
-0.05 \\
\end{bmatrix} \]

\[
eigenvals(A + M \cdot C) = \begin{bmatrix}
-0.545 \\
-0.259 + 0.547i \\
-0.259 + 0.547i \\
0.282 - 0.413i \\
0.282 + 0.413i \\
\end{bmatrix}
\]

\[
M \cdot C = \begin{bmatrix}
-0.5 & 0 & 0 & 0 \\
-0.3 & 0 & 0 & 0 \\
-0.1 & 0 & 0 & 0 \\
-0.05 & 0 & 0 & 0 \\
-0.05 & 0 & 0 & 0 \\
\end{bmatrix}
\]

The state update equations are:

\[
k = 0 \ldots 49
\]

\[
Xsys^{<k+1>} = A \cdot Xsys^{<k>} + B \cdot u_k
\]

\[
Ysys^{<k>} = C \cdot Xsys^{<k>}
\]

\[
Xobs^{<k+1>} = A \cdot Xobs^{<k>} + B \cdot u_k - M \cdot C \cdot (Xsys^{<k)} - Xobs^{<k>})
\]

---

**Vehicles**

Sample, k

- Xobs1
- Xobs2
- Xobs3
- xobs4
- Xobs5
The state update equations are:

\[
k = 0, 0.49
\]

\[
X_{sys}^{<k+1>} = A \cdot X_{sys}^{<k>} + B \cdot u_k
\]

\[
Y_{sys}^{<k>} = C \cdot X_{sys}^{<k>}
\]

\[
X_{obs}^{<k+1>} = A \cdot X_{obs}^{<k>} + B \cdot u_k - M \cdot C \cdot (X_{sys}^{<k>} - X_{obs}^{<k>})
\]

---

c)

\[
M = \begin{bmatrix}
-0.05 \\
-0.1 \\
-0.2 \\
-0.3 \\
-0.35 \\
\end{bmatrix}
\]

\[
eigenvals(A + M \cdot C) = \begin{bmatrix}
0.65 - 0.635i \\
0.65 + 0.635i \\
-0.308 - 0.695i \\
-0.308 + 0.695i \\
-0.734
\end{bmatrix}
\]

\[
M \cdot C = \begin{bmatrix}
-0.05 & 0 & 0 & 0 & 0 \\
-0.1 & 0 & 0 & 0 & 0 \\
-0.2 & 0 & 0 & 0 & 0 \\
-0.3 & 0 & 0 & 0 & 0 \\
-0.35 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Vehicles

Sample, k

Vehicles

Sample, k
Example 3.2:

Input Sequences

\[
\begin{align*}
k &:= 0..14 \\
u_k &:= 30 \\
h &:= 0..29 \\
k &:= 15..19 \\
u_k &:= 30 + 4 \cdot (k - 14) \\
h &:= 30..39 \\
k &:= 20..24 \\
u_k &:= 50 \\
h &:= 40..49 \\
k &:= 25..29 \\
u_k &:= 50 - 6 \cdot (k - 24) \\
h &:= 50..59 \\
k &:= 30..39 \\
u_k &:= 20 \\
h &:= 60..79 \\
j &:= 0..149 \\
u_{2,j} &:= 3 \\
j &:= 150..199 \\
u_{2,j} &:= 3 + .04 \cdot (j - 149) \\
j &:= 200..249 \\
u_{2,j} &:= 5 \\
j &:= 250..299 \\
u_{2,j} &:= 5 - .06 \cdot (j - 249) \\
j &:= 300..399 \\
u_{2,j} &:= 2
\end{align*}
\]

The initial conditions:

\[
\begin{align*}
X_{sys}^{<0>} &:= \begin{bmatrix} 30 \\ 30 \\ 30 \\ 30 \end{bmatrix} \\
X_{obs}^{<0>} &:= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{align*}
\]
The PLANT matrix, $A$, is defined as:

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The INPUT matrix, $B$, is defined as:  

$$B = \begin{bmatrix} \end{bmatrix}$$

The OUTPUT matrix, $C$, is defined as:  

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} -.2 \\ -2 \\ -.2 \\ -.2 \end{bmatrix}$$

$$\text{eigenvals}(A + M \cdot C) = \begin{bmatrix} 0.518 - 0.594i \\ 0.518 + 0.594i \\ -0.282 - 0.633i \\ -0.282 + 0.633i \\ -0.671 \end{bmatrix}$$

$$M \cdot C = \begin{bmatrix} -0.2 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 \end{bmatrix}$$

The state update equations are:

$$k := 0..39$$

$$X_{\text{sys}}^{<k+1>} = A \cdot X_{\text{sys}}^{<k>} + B \cdot u_k$$

$$Y_{\text{sys}}^{<k>} = C \cdot X_{\text{sys}}^{<k>}$$

$$X_{\text{obs}}^{<k+1>} = A \cdot X_{\text{obs}}^{<k>} + B \cdot u_k - M \cdot C \cdot (X_{\text{sys}}^{<k>} - X_{\text{obs}}^{<k>})$$
Vehicles

\[ X_{obs1} \]
\[ X_{obs2} \]
\[ X_{obs3} \]
\[ x_{obs4} \]
\[ X_{obs5} \]

Sample, k

Vehicles

\[ X_{sys1} - X_{obs1} \]
\[ X_{sys2} - X_{obs2} \]
\[ X_{sys3} - X_{obs3} \]
\[ X_{sys4} - X_{obs4} \]
\[ X_{sys5} - X_{obs5} \]

Sample, k
\( k : = 0 \ldots 79 \)

The PLANT matrix, \( A \), is defined as:

\[
A = \begin{bmatrix}
0.5 & 0 & 0 & 0 \\
0 & 0.5 & 0 & 0 \\
0 & 0 & 0.5 & 0 \\
0 & 0 & 0 & 0.5
\end{bmatrix}
\]

The INPUT matrix, \( B \), is defined as:

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

The OUTPUT matrix, \( C \), is defined as:

\[
C = \begin{bmatrix}
0.5 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The state update equations are:

\[
k : = 0 \ldots 79
\]

\[
X_{sys}^{\langle k+1 \rangle} = A \cdot X_{sys}^{\langle k \rangle} + B \cdot u_k
\]

\[
Y_{sys}^{\langle k \rangle} = C \cdot X_{sys}^{\langle k \rangle}
\]

\[
X_{obs}^{\langle k+1 \rangle} = A \cdot X_{obs}^{\langle k \rangle} + B \cdot u_k - M \cdot C \cdot (X_{sys}^{\langle k \rangle} - X_{obs}^{\langle k \rangle})
\]
b) 

\[ k = 0 \ldots 399 \]

The INPUT matrix, \( B \), is defined as:

\[
B = \begin{bmatrix}
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

The OUPUT matrix, \( C \), is defined as:

\[
C = \begin{bmatrix}
0.1 & 0.0 & 0.0 & 0.0 \\
-0.943 & -0.1 & 0.0 & 0.0 \\
0.943 & 0.1 & 0.0 & 0.0 \\
0.094 & 0.0 & 0.0 & 0.0 \\
\end{bmatrix}
\]

The state update equations are:

\[
k = 0 \ldots 399
\]

\[
X_{sys}^{k+1} = A \cdot X_{sys}^k + B \cdot u_2^k
\]

\[
Y_{sys}^k = C \cdot X_{sys}^k
\]

\[
X_{obs}^{k+1} = A \cdot X_{obs}^k + B \cdot u_2^k - M \cdot C \cdot (X_{sys}^k - X_{obs}^k)
\]
Example 3.3:

Input Sequence:

\[ k = 0..14 \]
\[ u_k = 30 \]
\[ k = 15..19 \]
\[ u_k = 30 + 4 \cdot (k - 14) \]
\[ k = 20..24 \]
\[ u_k = 50 \]
\[ k = 25..29 \]
\[ u_k = 50 - 6 \cdot (k - 24) \]
\[ k = 30..79 \]
\[ u_k = 20 \]
\[ k = 0..79 \]

a)

The two PLANT matrices are defined as:

\[
\begin{bmatrix}
0.1 & 0 & 0 & 0 \\
0 & 0.1 & 0 & 0 \\
0 & 0 & 0.1 & 0 \\
0 & 0 & 0 & 0.1
\end{bmatrix}
\quad \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\quad \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

The INPUT matrix is the same for both and is defined as:

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix}
\]

The OUTPUT matrices are defined as:

\[
C_{sys} = (0.9, 0, 0, 0, 0) \quad C_{obs} = (1, 0, 0, 0, 0)
\]

The initial conditions are, as defined above,

\[
X_{sys}^{<0>} = \begin{bmatrix}
30 \\
30 \\
30 \\
30
\end{bmatrix} \quad X_{obs}^{<0>} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
\[ M := \begin{bmatrix} -0.2 \\ -0.2 \\ -0.2 \\ -0.2 \end{bmatrix} \]

\[ \text{eigenvals}(A_{obs} + M \cdot C_{obs}) = \begin{bmatrix} 0.518 - 0.594i \\ 0.518 + 0.594i \\ -0.282 - 0.633i \\ -0.282 + 0.633i \\ -0.671 \end{bmatrix} \]

\[ M \cdot C_{obs} = \begin{bmatrix} -0.2 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \\ -0.2 & 0 & 0 & 0 & 0 \end{bmatrix} \]

The state update equations are:

\[
k = 0, 79
\]

\[
X_{sys}^{<k+1>} := A_{sys} \cdot X_{sys}^{<k>} + B \cdot u_k
\]

\[
Y_{sys}^{<k>} := C_{sys} \cdot X_{sys}^{<k>}
\]

\[
X_{obs}^{<k+1>} := A_{obs} \cdot X_{obs}^{<k>} + B \cdot u_k - M \left( C_{sys} \cdot X_{sys}^{<k>} - C_{obs} \cdot X_{obs}^{<k>} \right)
\]
Figure 3.11a

Figure 3.11b

Figure 3.11c

Figure 3.11d

Figure 3.11e
The state update equations are used to calculate the new results:

\[ k = 0 \ldots 79 \]

\[ X_{sys}^{k+1} = A_{sys} \cdot X_{sys}^k + B \cdot u_k \]

\[ Y_{sys}^k = C_{sys} \cdot X_{sys}^k \]

\[ X_{obs}^{k+1} = A_{obs} \cdot X_{obs}^k + B \cdot u_k - M \left( C_{sys} \cdot X_{sys}^k - C_{obs} \cdot X_{obs}^k \right) \]

\[ A_{sys} = \begin{bmatrix} 0.25 & 0.75 & 0 & 0 & 0 \\ 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0 & 0.25 & 0.75 & 0 \\ 0 & 0 & 0 & 0.25 & 0 \end{bmatrix} \]

\[ C_{sys} = (0.75, 0, 0, 0, 0) \]
The state update equations are used to calculate the new results:

\[
X_{sys}^{k+1} = Asys \cdot X_{sys}^{k} + B \cdot u_k \\
Y_{sys}^{k} = Csys \cdot X_{sys}^{k} \\
X_{obs}^{k+1} = Aobs \cdot X_{obs}^{k} + B \cdot u_k - M \cdot (Csys \cdot X_{sys}^{k} - Cobs \cdot X_{obs}^{k})
\]
Figure 3.15 a

Figure 3.15 b

Figure 3.15 c

Figure 3.15 d

Figure 3.15 e
$X_{sys1}$-$X_{obs1}$
$X_{sys2}$-$X_{obs2}$
$X_{sys3}$-$X_{obs3}$
$X_{sys4}$-$X_{obs4}$
$X_{sys5}$-$X_{obs5}$
BIBLIOGRAPHY


