Binomial Tree Model: Pricing European and American Stock Options

University of Dayton

Jacob Pieniazek

Advisor: Dan Ren, Ph.D.

The Purpose of this Project

To understand the mechanism by which to price European stock options and extrapolate this knowledge to calculating the more complex American options and understanding the behavior of the boundary upon which the American option will be exercised—the optimal exercise boundary. Utilizing Python programming software and a theoretical understanding of stock option price modeling, we were able to effectively create an algorithm to calculate the stock-price binomial tree & price both European and American options.

Model Parameters:

Initial Stock Price (S) = \$11

Exercise Price (K) = \$12

Risk-Free Rate (r) = 0.10

Number of Steps (N) = 6

Up-Factor (u) = 1.25

Down-Factor (d) = 0.8

Model Assumptions:

1) Arbitrage Free - $d < 1 + r < u$
2) $ud = 1$

Understanding Call & Put Options

A **call/put** option is a contract between a seller and a buyer so that the buyer has the right, but not the obligation, to **buy/sell** an underlying asset at a certain time for a certain agreed-upon price **K**.

Payoffs at time N

$$Call \Rightarrow V(N,j) = (S_{N,j} - K)^{+} = \max\{S_{N,j} - K, 0\} = \begin{cases} S_{N,j} - K, & S_{N,j} > K \\ 0, & S_{N,j} \leq K \end{cases}$$

$$Put \Rightarrow V(N,j) = (K - S_{N,j})^{+} = \max\{K - S_{N,j}, 0\} = \begin{cases} K - S_{N,j}, & S_{N,j} < K \\ 0, & S_{N,j} \geq K \end{cases}$$

European & American Options

A **European Option** is a contract between a seller and a buyer so that the buyer has the right to exercise the option ONLY at the expiration date.

An **American Option** is a contract between a seller and a buyer so that the buyer has the right to exercise the option at ANY time up to expiration date.

Method of Pricing Options

European

I. Calculate the stock price binomial tree for N steps
II. Determine which nodes at the terminal time N in the stock price binomial tree the option will be exercised
III. Discount the option price from time N-1 to time 0 for each node

$$V(i,j) = \left(\frac{\tilde{p}}{r}\right) V(i+1, j+1) + \left(\frac{\tilde{q}}{r}\right) V(i+1, j)$$

American

I. Calculate the stock price binomial tree for N steps
II. Determine which nodes at the terminal time N in the stock price binomial tree the option will be exercised
III. Determine whether to discount or exercise from time N-1 to time 0 for each node

$$V(i,j) = \max\left[\left(\frac{\tilde{p}}{r}\right) V(i+1, j+1) + \left(\frac{\tilde{q}}{r}\right) V(i+1, j),\ K - Sd^{j}u^{i-j}\right]$$

Notes: Where j is the number of down-steps, $(i-j)$ is the number of up-steps, r is the risk-free rate, $\tilde{p} = \frac{(1+r)-d}{u-d}$, $\tilde{q} = 1 - \tilde{p}$, u is the up-factor, d is the down-factor, S is the initial stock price, K is the exercise price.

Stock Price Binomial Tree (*N*)

j \ *i*	0	1	2	3	4	5	6
0	11	13.75	17.1875	21.48438	26.85547	33.56934	41.96167
1	0	8.8	11	13.75	17.1875	21.48438	26.85547
2	0	0	7.04	8.8	11	13.75	17.1875
3	0	0	0	5.632	7.04	8.8	11
4	0	0	0	0	4.5056	5.632	7.04
5	0	0	0	0	0	3.60448	4.5056
6	0	0	0	0	0	0	2.88358

European/American Call Binomial Tree (*N*)

j \ *i*	0	1	2	3	4	5	6
0	4.657333	6.547043	9.100621	12.49642	16.93811	22.66025	29.96167
1	0	2.275114	3.404	5.039201	7.361972	10.57528	14.85547
2	0	0	0.699878	1.154799	1.905418	3.143939	5.1875
3	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0

European Put Binomial Tree (*N*)

j \ *i*	0	1	2	3	4	5	6
0	0.431021	0.248099	0.109282	0.027826	0	0	0
1	0	0.92617	0.600161	0.304978	0.091827	0	0
2	0	0	1.856039	1.370576	0.822773	0.30303	0
3	0	0	0	3.383778	2.877355	2.109091	1
4	0	0	0	0	5.411755	5.277091	4.96
5	0	0	0	0	0	7.304611	7.4944
6	0	0	0	0	0	0	9.116416

American Put Binomial Tree (*N*)

j \ *i*	0	1	2	3	4	5	6
0	1.244173	0.452885	0.139638	0.027826	0	0	0
1	0	3.2	1.215245	0.405153	0.091827	0	0
2	0	0	4.96	3.2	1.153352	0.30303	0
3	0	0	0	6.368	4.96	3.2	1
4	0	0	0	0	7.4944	6.368	4.96
5	0	0	0	0	0	8.39552	7.4944
6	0	0	0	0	0	0	9.116416

Notes: Where j is the number of down-steps, $(i-j)$ is the number of up-steps, $\max(i) = \max(j) = N$. Highlighted Region is optimal exercise boundary. European Call = American Call ALWAYS. American Put > European Put USUALLY.