Riemann Sum Construction to Obtain the Power Rule for Integral Calculus

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Introduction

- This poster verifies the power rule for integral calculus using Riemann Sums with geometric progressions
- This poster will show that if $k = \frac{u}{v}$, $k \neq -1$ is rational and $0 < a < b$ then,

$$\int_a^b x^k dx = \frac{b^{k+1} - a^{k+1}}{k+1}$$

Geometric Series Verification

$$\sum_{k=0}^{n-1} r^k = 1 + r + r^2 + \cdots + r^{n-1} = \frac{r^n - 1}{r-1}$$

$$s_n = 1 + r^1 + r^2 + \cdots + r^{n-1}$$

$$rs_n = r^1 + r^2 + r^3 + \cdots + r^n$$

$$rs_n - s_n = r^n - 1$$

$$s_n(r-1) = r^n - 1$$

$$s_n = \frac{r^n - 1}{r-1}$$

Geometric Progression

- The calculation will use a partition of the interval $[a, b]$ that is a geometric progression (see Figure 2)
- In MTH 168 integrals of x^k where k is a positive integer are found using an algebraic progression (see Figure 1)
- Let n be a positive integer and set $q = (\frac{b}{a})^{\frac{1}{n}}$
- The geometric progression is found to be

$$P = \{a, a\left(\frac{b}{a}\right)^{\frac{1}{n}}, a\left(\frac{b}{a}\right)^{\frac{2}{n}}, a\left(\frac{b}{a}\right)^{\frac{3}{n}}, \ldots, b\}$$

$$\Delta x_i = a\left(\frac{b}{a}\right)^{\frac{i}{n}} - a\left(\frac{b}{a}\right)^{\frac{i-1}{n}} = aq^{i-1}(q-1)$$

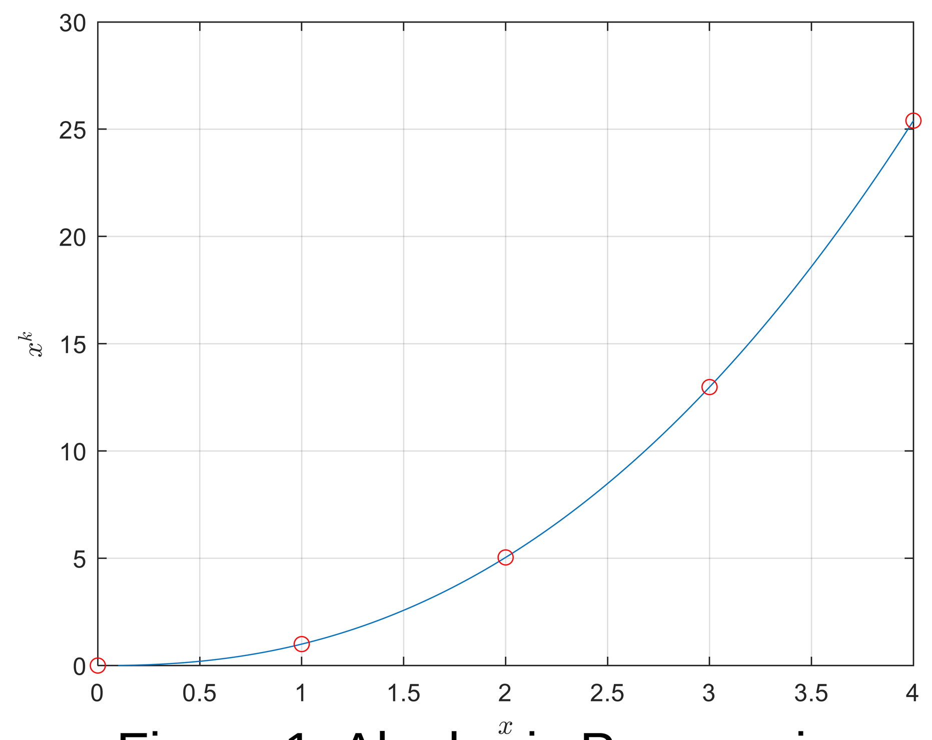

Figure 1: Algebraic Progression

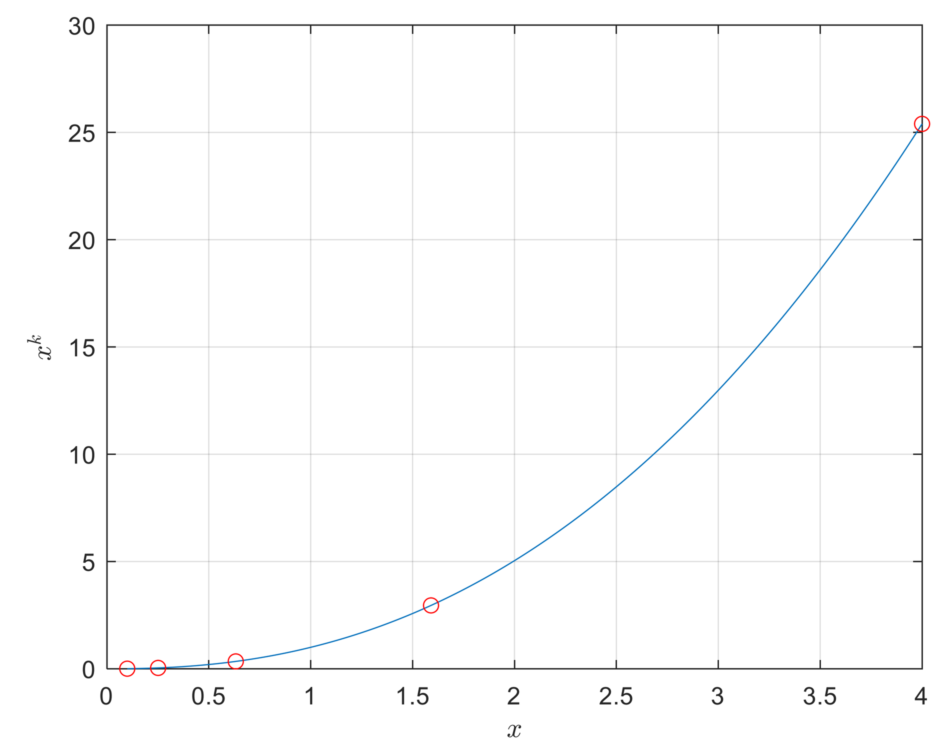

Figure 2: Geometric Progression

Lower Riemann Sum Construction

$$s_n = \sum_{i=1}^{n} (x_{i-1}^k)(\Delta x_i) \qquad s_n = \sum_{i=1}^{n} a^k q^{(i-1)k}(aq^{i-1}(q-1))$$

$$= a^{k+1}(q-1)\sum_{i=1}^{n} q^{(i-1)(k+1)} \qquad = a^{k+1}(q-1)\sum_{i=0}^{n-1}(q^{k+1})^i$$

$$= a^{k+1}(q-1)\frac{(q^{k+1})^n - 1}{q^{k+1}-1} \qquad = a^{k+1}(q-1)\frac{(\frac{b}{a})^{(k+1)\frac{n}{n}} - 1}{q^{k+1}-1}$$

$$= a^{k+1}(q-1)\frac{(\frac{b}{a})^{(k+1)} - 1}{q^{k+1}-1} \qquad = (q-1)\frac{b^{k+1} - a^{k+1}}{q^{k+1}-1}$$

Upper Riemann Sum Construction

$$t = \sum_{i=1}^{n} (x_i^k)(\Delta x_i) \qquad t_n = \sum_{i=1}^{n} a^k q^{(i)k}(aq^{i-1}(q-1))$$

$$= a^{k+1}q(q-1)\sum_{i=1}^{n} q^{(i-1)(k+1)} \qquad = a^{k+1}q(q-1)\sum_{i=0}^{n-1}(q^{k+1})^i$$

$$= qs_n \qquad = q(q-1)\frac{b^{k+1} - a^{k+1}}{q^{k+1}-1}$$

Limit as $n \to \infty$

$$\lim_{n\to\infty} q^i = \lim_{n\to\infty}((\frac{b}{a})^i)^{\frac{1}{n}}$$

- Assume k is a positive integer

$$N = \frac{q^{k+1}-1}{q-1} = \frac{(q-1)\sum_{i=0}^{k} q^i}{q-1} \qquad \lim_{n\to\infty} \sum_{i=0}^{k} q^i = k+1$$

- Assume k is a negative integer, $l = -k$

$$= \frac{q^{k+1}(1 - q^{-k-1})}{q-1} = \frac{-q^{k+1}(q^{l-1} - 1)}{q-1}$$

$$\lim_{n\to\infty} -q^{k+1}\sum_{i=1}^{l-2} q^i = -(l-1) = k+1$$

- Assume $k = \frac{u}{v}$ and set $r = q^{\frac{1}{v}}$

$$\frac{q^{k+1}-1}{q-1} = \frac{q^{\frac{u}{v}+1}-1}{q-1} = \frac{q^{\frac{u+v}{v}}-1}{q-1} = \frac{r^{u+v}-1}{r^v-1}$$

$$\lim_{n\to\infty} \frac{\frac{r^{u+v}-1}{r-1}}{\frac{r^v-1}{r-1}} = \frac{u+v}{v} = \frac{u}{v} + 1 = k+1$$