Two measures of non-planarity of graphs

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Planar Graphs

A planar graph is a graph that can be drawn in the plane with no edges crossing. A planar graph is called maximal planar if adding an edge between any two nonadjacent vertices result in a nonplanar graph.

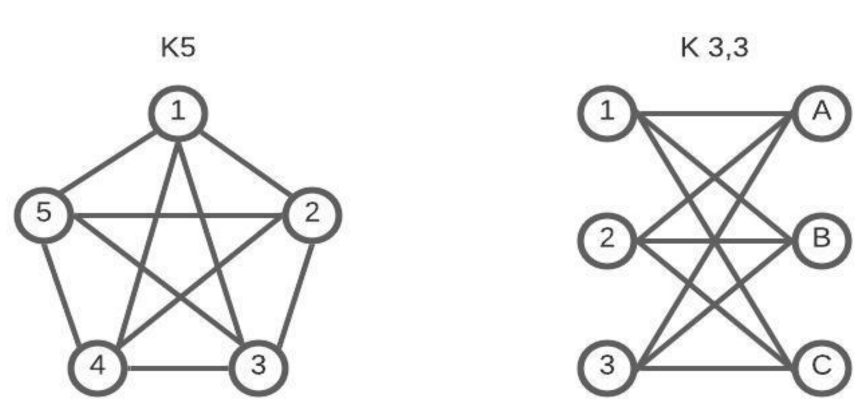
Theorem: If G is a maximal planar graph with p vertices and q edges, $p \geq 3$, then $q = 3p - 6$.

Nonplanar Graphs

The most famous nonplanar graphs are K_5, the complete graph on 5 vertices, and $K_{3,3}$, the complete bipartite graph on 6 vertices with two equal partite sets. K_5 has 5 vertices and 10 edges. A planar graph on 5 vertices can have at most 9 edges. Hence, K_5 is nonplanar.

Euler's Polyhedral Formula: If a planar drawing of a connected graph with p vertices and q edges has r regions, the $p - q + r = 2$.

Suppose $K_{3,3}$ is planar. Then r = 5. $K_{3,3}$ is bipartite so every region has an even number of sides, which are edges of the graph. Thus, every region must have at least 4 sides. However, each edge is counted twice since it is part of the boundary of two regions. So $2q \geq 5 \times 4$, or 20. But $K_{3,3}$ has 9 edges, so 2q is only 18. We have a contradiction, so $K_{3,3}$ is nonplanar.

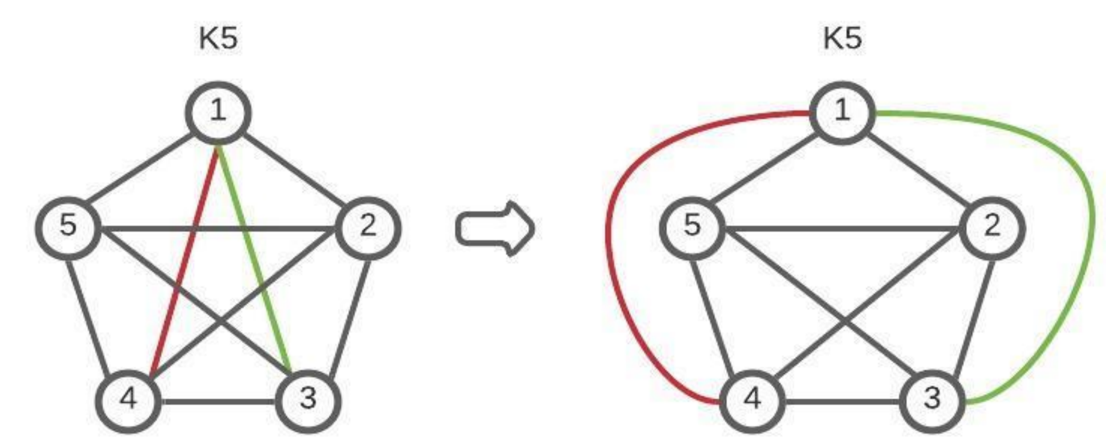


Kuratowski's Theorem

A graph G is planar if and only if G does not contain a subdivision of K_5 or $K_{3,3}$ as a subgraph. A graph G' is called a subdivision of graph G if G' = G or one or more vertices of degree 2 are inserted into one or more edges of G.

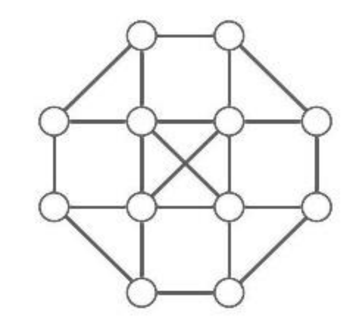
Crossing Number

For any given graph G, the crossing number is the minimum number of times edges in G cross each other. We know that both K_5 and $K_{3,3}$ are nonplanar. What is each graph's crossing number? From the drawing above, K_5's edges cross 5 times, but is that the minimum number of times edges in K_5 have to cross?



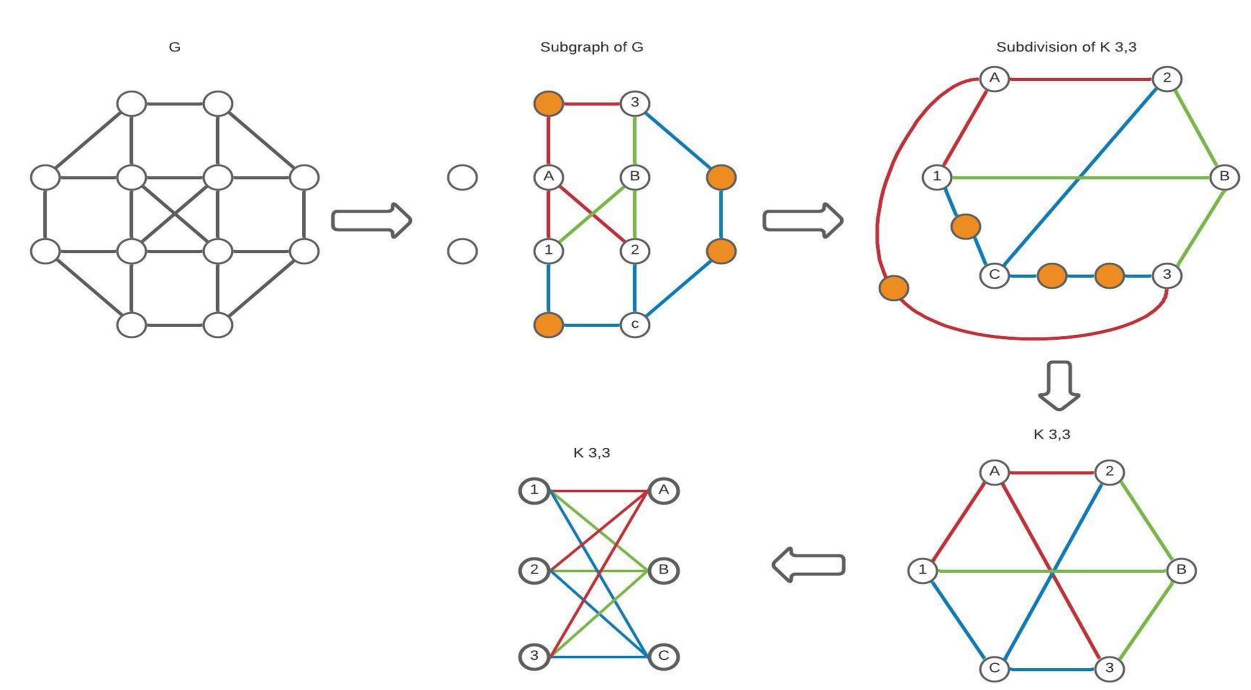
As shown, K_5's crossing number is 1, denoted cr(K_5) = 1.

Example:



What is the crossing number for this graph?

The drawing shows 1 crossing. Is there a planar drawing of this graph? This is a graph with 12 vertices and 22 edges. The only way this graph is nonplanar and has a crossing number of 1 is if it has a subdivision of K_5 or $K_{3,3}$ according to Kuratowski's theorem.



Note that the yellow vertices are the inserted vertices of degree 2 according to the definition of subdivision.

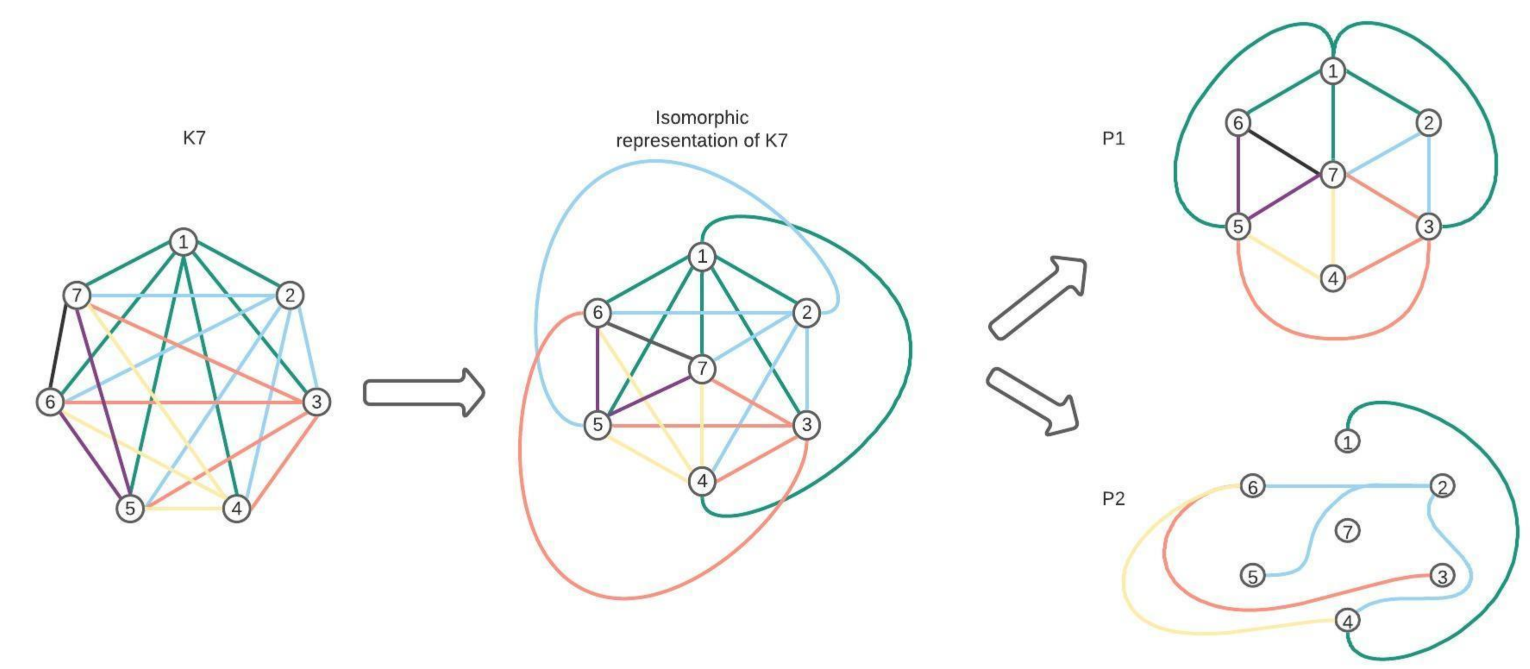
Thickness Number

Another measure of nonplanarity is thickness number. The thickness of a graph G, denoted by Θ(G), is the minimum number of planar subgraphs in a decomposition of G into planar subgraphs. We show that the thickness number of K_7 is 2.

According to Richard Guy's inequality for the complete graph,

$\text{cr}(K_n) \leq \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$. The crossing number for K_7 is less than or equal to 9.

We will work with an isomorphic representation of K_7.



Clearly, both P1 and P2 are planar, and $|E(P1)| + |E(P2)| = 15 + 6 = 21 = |E(K_7)|$, therefore Θ($K_7$) = 2.

We know a theorem for the thickness number of complete graphs.

$$\Theta(K_n) = \begin{cases} \lfloor\frac{n+7}{6}\rfloor \text{ if } n \neq 9,10 \\ 3 \text{ if } n = 9,10 \end{cases}$$

This confirms that Θ(K_7) = $\lfloor\frac{7+7}{6}\rfloor$ = 2.

Future Work

According to Guy's formula, K_n, for 9 ≤ n ≤ 16, has thickness number 3. K_{17} has 17 vertices and 136 edges. That is exactly 3 maximal planar graphs plus 1 extra edge. It makes sense that K_{17} has thickness number 4.

But K_9 has 9 vertices and 36 edges. A maximal planar graph on 9 vertices has 21 edges. If we form a maximal planar graph on 9 vertices, we will have 15 edges left over. Since we know that the thickness number of K_9 is 3, this must mean that there is a subdivision of either K_5 or $K_{3,3}$ in those 15 edges, no matter which edges we choose to form the planar graph.

Bibliography

Hartsfield, Nora & Ringel, Gerhard (2003). Pearls in Graph theory A comprehensive Introduction. Mineola, NY: Dover publications, Inc.

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