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Mathematics Capstone
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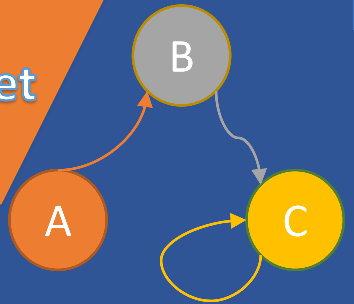
A Mathematical Exploration of Mad Vet Scenarios



1 One Input, At Least One Output

- Each machine takes a single animal of a given species as input and outputs a nonempty collection of animals from any number of species.

2 Reversibility

- Each machine can operate forwards or backwards and may operate as many times as there are available inputs.

3 A Machine For Each Species

- There is a one-to-one correspondence between the species and the machines.

Menageries — Animal Collection

- $\bar{x} = (x_1, x_2, .., x_n)$, where $\bar{x}$ represents the number of animals of species $1 \leq i \leq n$

Equivalence — Transformation Sequence

- Let S be the set of menageries for a given scenario
- Let $\bar{x}, \bar{y} \in S$, so $\bar{x} = (x_1, x_2, .., x_n), \bar{y} = (y_1, y_2, \dots, y_n)$
- $\bar{x}$ is related to $\bar{y}$, $\bar{x}\ R\ \bar{y}$, if and only if a sequence of one or more machines that transforms $\bar{x}$ into $\bar{y}$
- We take $\bar{x} \sim \{x_i A\} \sim \bar{y}$ to mean $\bar{x}$ is transformed into $\bar{y}$ via the application of machine A a total of x_i times
- This constitutes an equivalence relation on S
- The equivalence class of $\bar{x}$ in S is $[\bar{x}] = \{\bar{y} \in S : \bar{y} \sim \bar{x}\}$

Monoid — Transformation Analysis

Let $\bar{x} = (x_1, x_2, x_3) \in S$. Observe that

$$(x_1, x_2, x_3) \sim \{x_1 A\} \sim (0, x_2 + x_1, x_3)$$
$$\sim \{(x_2 + x_1)B\} \sim (0, 0, x_1 + x_2 + x_3)$$

As $(0,0,0) \notin S$, we have $a + b + c \in \mathbb{Z}^+$. We have $(0,0,i) \ncong (0,0,j)$ for all $i \neq j$. Any given starting menagerie is indexed to some positive integer, and the given scenario is isomorphic to $\mathbb{Z}^+$. For example, $(3,2,0) \sim (0,0,5) \sim (2,3,0) \cong 5$, but $(3,1,0) \sim (0,0,4) \sim (1,3,0) \cong 4$.

Group — Mad Vet Group Identification Theorem

Construct the matrix H_Γ, representing machine output for the given scenario, where entry $u_{i,j}$ corresponds to the number of animals of species j produced by machine i

$$H_\Gamma = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}$$

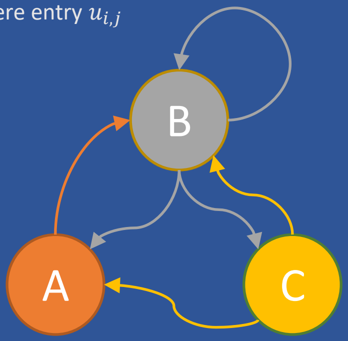
Combine with the identity matrix, representing machine input

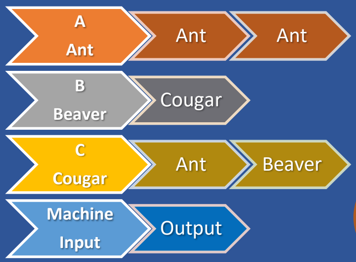
$$I_3 - H_\Gamma = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & -1 \\ -1 & -1 & 1 \end{pmatrix}$$

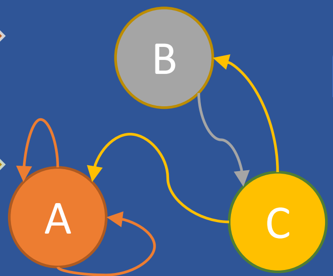
Convert to Smith normal form

$$I_3 - H_\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

It can be shown that the semigroup for this scenario is a group. By the Mad Vet Group Identification Theorem, that group is isomorphic to $\mathbb{Z}_1 \oplus \mathbb{Z}_1 \oplus \mathbb{Z}_3 \cong \{0\} \oplus \{0\} \oplus \mathbb{Z}_3 \cong \mathbb{Z}_3$







Semigroup — Mad Vet Group Test

Vertex A does not connect to cycle $e_{BC}e_{CB}$ so the graph is not cofinal. Additionally, the cycle at A has no exit. The graph fails the Mad Vet Group Test. Consequently, the associated semigroup is not a group.

Semigroup — Associative

- Let S be a set and let $*$ be a binary operation.
- $(S,*)$ is a semigroup if for all $u, v, w \in S$, $u * (v * w) = (u * v) * w$

Monoid — Identity

- $(S,*)$ is a monoid if it is a semigroup and contains an identity element, that is , there exists $I \in S$ such that for all $u \in S$, $u * I = u$

Group — Inverse

- $(S,*)$ is a group if it is a monoid and every element in S has an inverse. That is, for all $u \in S$, there exists $v \in S$ such that $u * v = I$

Mad Vet Semigroup

- The set of all equivalence classes W for a given scenario forms a semigroup.
- Addition on W is well-defined as $[\bar{x}] + [\bar{y}] = [\bar{x} + \bar{y}]$
- Associativity inherited from vector addition.

Mad Vet Group Test

- The Mad Vet semigroup W of a Mad Vet scenario is a group if and only if the corresponding Mad Vet graph Γ has the following properties

1. Γ is cofinal, that is, every vertex of Γ connects to every cycle and every sink
2. Every cycle in Γ has an exit, an edge adjacent to a vertex in a cycle that leads away from the cycle

Mad Vet Group Identification Theorem

- Given a Mad Vet Scenario whose Mad Vet semigroup, W, is a group, let Γ be its associated graph. Then W is isomorphic to $\mathbb{Z}_{\alpha_1} \oplus \mathbb{Z}_{\alpha_2} \oplus \dots \oplus \mathbb{Z}_{\alpha_q} \oplus \mathbb{Z}^{n-q}$, where $\alpha_1, \alpha_2, \dots, \alpha_q$ are the nonzero diagonal entries of the Smith normal form of the matrix $I_n - H_\Gamma$

Sources

1. Gene Abrams & Jessica K. Sklar (2010) The Graph Menagerie: Abstract Algebra and the Mad Veterinarian, Mathematics Magazine, 83:3, 168-179, DOI: 10.4169/002557010X494814
2. Ara, Pere & Moreno-Frías, M. & Pardo, Enrique. (2005). Nonstable K-theory for Graph Algebras. Algebras and Representation Theory. 10. 10.1007/s10468-006-9044-z.