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Sequential Order of Compact Scattered Spaces

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Introduction

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For any subset A of a space X, $A^{(1)}$ is the "Frechet" closure of A, i.e. $A^{(1)} = \{x \in X : (\exists \{a_n\}_n \subset A) \ \ \{a_n\} \to x\}$ and this is step 1 of the sequential closure hiearchy where $A^{(0)} = A$, and $A^{(\alpha)} = \left(\bigcup_{\beta<\alpha} A^{(\beta)}\right)^{(1)}$ naturally $A^{(\omega_1+1)} = \bigcup_{\alpha<\omega_1} A^{(\alpha)}$

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X is sequential if $A^{(\omega_1)}$ is closed for all $A \subset X$

Sequential order and Bashkirov problem

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What is the minimal upper bound on the sequential order of all compact sequential spaces? (wlog compactifications of ω)

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3. MA implies that it is at least 4.

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and satisfied that $\omega^{(\alpha)} = \rho^{-1}([0, \alpha])$ which PFA bounds at $\omega + 1$

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Lemma (PFA, Step 1)

If X is compact and sequential, and $z \in \overline{D} \setminus D^{(1)}$ for some $D \subset X$, then there is a sequence $\{x_\alpha : \alpha \in \omega_1\} \subset D^{(1)}$ that co-countably converges to z.

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Assumption going forward

X is compact, scattered, and sequential. ρ denotes the scattering level, $\{W_x : x \in X\}$ the ρ-nbd assignment

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3. $\{x(n, \sigma, m, \alpha) : \alpha \in \omega_1\}$ will denote a sequence from $(D(n, \sigma))^{(1)}$ co-countably converging to $w(n, \sigma, m)$

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then, for almost all α, there is a point $v_f(\sigma,\alpha) \neq v(\sigma,\alpha)$ because $x(n,\sigma,\alpha,f(n)) \in W_{w_f} \cap W_{v(\sigma,\alpha)}$, hence $\rho(v_f(\sigma,\alpha)) \geq \omega$

Summary

We were able to “raise” $\rho(v(\sigma, \alpha))$ because each $y(n, \sigma, \alpha)$ was a limit of the sequence $\{x(n, \sigma, \alpha, m) : m \in \omega\}$ or more clearly, $v(\sigma, \alpha)$ arose from the supporting σ-enumerated structure

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so IF each $x(n,\sigma,\alpha,m)$ was itself some $y(n,\tau,\beta)$ for a τ-structure, then $v_f(\sigma,\alpha)$ would itself be some $v(\tau,\beta)$

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First step is to realize that we have the option of choosing any sequence $\vec{\sigma} = \langle \sigma_n : n \in \omega \rangle$ rather than just a single constant σ, and using each $\{y(n,\sigma_n,\alpha) : n \in \omega\}$ to get $\{v(\vec{\sigma},\alpha) : \alpha \in \omega_1\}$.

the general $\omega^{<\omega}$-structure

Lemma

Since w_n *is the co-countable limit of* $\{y(n,\sigma,\alpha) : \alpha \in \omega_1\}$, *there is a cub* $C_\sigma \subset \omega_1$ *such that, for each* $\gamma \in C$, w_n *is in the closure of* $D(n,\sigma^\frown\gamma) = \{y(n,\sigma,\alpha) : \gamma \leq \alpha < \min(C_\sigma \setminus (\gamma+1))$.

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Lemma

We can then repeat ad infinitum and more generally construct for all n and all $\tau \in \omega_1^{<\omega}$
$D(n,\tau), \{w(n,\tau,m)\}_{m\in\omega}, \{x(n,\tau,m,\alpha)\}_{\alpha\in\omega_1}, \{y(n,\tau,\alpha)\}_{\alpha\in\omega_1}$
with the property that $\sigma \subsetneq \tau$, $D(n,\tau) \subset \overline{y(n,\sigma,\alpha) : \alpha \in \omega_1\}}$

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the contradiction to countable

The ω_1-sequence of values of $\mu(\vec{\sigma}^\frown\gamma)$ $(\gamma \in \omega_1)$ should be greater than $\mu(\vec{\sigma})$ because of a suitable w_f giving $v_f(\vec{\sigma}^\frown\gamma) \in W_{v(\vec{\sigma}^\frown\gamma)}$ for suitably chosen $v(\vec{\sigma}^\frown\gamma)$ witnessing $\mu(\vec{\sigma}^\frown\gamma)$.