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Some New Completeness Properties in Topological Spaces

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This is a joint work with Professor Süleyman Önal.

I should note that this work has not been completed yet.

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- we will introduce certain completeness properties in topological spaces having a quasi-pair-base
- and investigate some properties of the topological spaces having such a base
- and also investigate which spaces have such a base.

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A Little Motivation

Čech-completeness

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A space is **Čech-complete** if it is homeomorphic to a G_δ-subset of a compact space.

It is well known that for metrizable spaces Čech-completeness is equivalent to complete metrizability.

One of the generalisations of completeness of metric spaces is subcompactness.

Definition (*Subcompact space*)

A space X is called **subcompact** if it has a base $\mathcal{B}$ of nonempty open subsets with the property that every regular open filter base $\mathcal{F}$ in $\mathcal{B}$ has nonempty intersection. Such a base $\mathcal{B}$ is called a subcompact base for X.

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Definition (*Regular filter base*)

A **regular open filter base** on a space X is nonempty collection of open sets $\mathcal{F}$ such that for any $G, H \in \mathcal{F}$ there is $F \in \mathcal{F}$ such that $\overline{F} \subseteq G \cap H$.

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So, for metrizable spaces, subcompactness is equivalent to complete metrizability.

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Since we are going to use a different characterization, let us not dwell on this definition of domain representability.

A Little Motivation

Domain Representability

Fortunately, Fleissner and Yengulalp gave a simplified characterization of domain representablity, in

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and then we have slightly changed it by using the downward directedness instead of upward directedness, and carriying out the required adjustments.

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D5-) if $F \subseteq P$ and $(F, \ll)$ is downward directed, then $\bigcap_{p \in F} \varphi(p) \neq \emptyset$.

In

[1] H. Bennett and D. Lutzer, *Domain Representable spaces*, Fund. Math.189 (3) (2006) 255-268

Bennett and Lutzer proved that Čech-complete spaces are domain representable.

(Unfortunately, it is not known yet whether Čech-complete spaces are subcompact or not.)

They also proved that subcompact regular spaces are domain representable, in

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Such a base $\mathcal{B}$ is called a **generalised subcompact base** for X.

Yengulalp proved that generalised subcompactness is equivalent to domain representability, in

[13] L. Yengulalp, *Coding strategies, the Choquet game, and domain representability*, Topol. Appl. 202 (2016) 384-396.

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Let $\mathcal{P}^{\star}(X)$ be the set of all nonempty subsets of the topological space X.

A **pair-collection** $\mathcal{P} = \{(A, B) : \overline{A} \subseteq B\}$ in X is a collection of subsets of $\mathcal{P}^{\star}(X) \times \mathcal{P}^{\star}(X)$ together with the partial order

$$\text{"}(A, B) \ll (C, D) \text{ if and only if } B \subseteq C.\text{"}$$

Let $\mathcal{P}$ be a pair-collection.

- A subset $\mathcal{F}$ of $\mathcal{P}$ is called a **filter base** in $\mathcal{P}$ if for any $(A, B), (C, D) \in \mathcal{F}$ there exists $(E, F) \in \mathcal{F}$ such that $(E, F) \ll (A, B)$ and $(E, F) \ll (C, D)$.

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- A nonempty subset $\mathcal{H}$ of $\mathcal{P}$ is said to have the **finite intersection property (f.i.p)** if $\bigcap_{(A,B)\in S} B \neq \emptyset$ for every finite subfamily $\mathcal{S}$ of $\mathcal{P}$.

Let $\mathcal{P}$ be a pair-collection.

Definition (*Complete pair-collection*)

$\mathcal{P}$ is **complete** if $\bigcap_{(A,B)\in\mathcal{F}} B \neq \emptyset$ for every filter base $\mathcal{F}$ in $\mathcal{P}$.

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Definition (*Countably complete pair-collection*)

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Definition (*L-complete pair-collection*)

$\mathcal{P}$ is *L*-**complete** if $\bigcap_{(A,B)\in\mathcal{F}} B \neq \emptyset$ for every filter base $\mathcal{F}$ in $\mathcal{P}$ whenever $\bigcap_{(A,B)\in\mathcal{C}} B \neq \emptyset$ for every countable subfamily $\mathcal{C}$ of $\mathcal{F}$.

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Definition (*fip-complete*)

$\mathcal{P}$ is **fip-complete** if $\bigcap_{(A,B)\in\mathcal{H}} B \neq \emptyset$ for every subfamily $\mathcal{H}$ of $\mathcal{P}$ having the finite intersection property.

We have the following relations between aforementioned completeness properties of pair-collections:

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countably complete + L-complete $\Rightarrow$ complete.

Definition (*Pair-base*)

Let $\mathcal{P} = \left\{(A, B) : \overline{A} \subseteq B\right\}$ be a pair-collection in the topological space X. $\mathcal{P}$ is called a **quasi-pair-base** for the space X if for every $x \in X$ and every open neighborhood U of x there is $(A, B) \in \mathcal{P}$ such that $x \in \overset{\circ}{A}$ and $B \subseteq U$. A quasi-pair-base $\mathcal{P}$ is called a **pair-base** for the space X if all A's are open.

We observe that if a space X has a complete quasi-pair base

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Hence we have that

the space X has a complete quasi-pair-base if and only if X has a complete pair-base.

MAIN RESULTS

Hereafter, all spaces are assumed to be regular.

MAIN RESULTS

Theorem

Subcompact spaces have a complete pair-base.

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Subcompact spaces have a complete pair-base.

Sketch of proof

Let X be a subcompact space and $\mathcal{B}$ is a subcompact base for X. Then

$$\mathcal{P} = \left\{ \left(U, V \right) : U, V \in \mathcal{B} \text{ and } \overline{U} \subseteq V \right\}$$

is a complete pair-base for X.

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Theorem

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Sketch of proof

Suppose $\mathcal{P}$ is a complete pair-base and $\mathcal{B}$ is any base for X. Define the relation $\prec$ on $\mathcal{B}$ by the rule:

$$U \prec V \text{ if and only if } U \subseteq A \subseteq B \subseteq V \text{ for a } (A, B) \in \mathcal{P},$$

for each $U, V \in \mathcal{B}$. Then $\mathcal{B}$ and $\prec$ satisfy $G1 - G5$, and hence $\mathcal{B}$ is generalised subcompact base for X.

MAIN RESULTS

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the property of having a complete pair-base lies somewhere between subcompactness and domain representability.

subcompactness $\Rightarrow$ having a complete pair-base $\Rightarrow$ domain representability

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The property of having a complete pair-base shows similarities with subcompactness and domain representability.

It is known that subcompactness is hereditary with respect to open subspaces, and domain representability is hereditary with respect to G_δ-subspaces, and both of them is closed under finite unions and arbitrary unions of open subspaces.

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Let us go through the outlines of proofs.

MAIN RESULTS

Theorem

If the topological space X has a complete pair-base then every open subspace of X has such a base.

Proof.

Let $\mathcal{P}$ be a complete pair-base for X, and let O be an open subspace of X. Define the family $\mathcal{Q} = \{(A, B) \in \mathcal{P} : B \subseteq O\}$. It is clear that the family $\mathcal{Q}$ is a pair-base for O. Since every filter base in $\mathcal{Q}$ is a filter base in $\mathcal{P}$, we have $\bigcap_{(A,B)\in\mathcal{F}} B \neq \emptyset$ for every filter base $\mathcal{F}$ in $\mathcal{Q}$. Hence $\mathcal{Q}$ is complete. □

Theorem

If a topological space has a complete pair-base, then every dense G_δ-subset of it has a complete pair-base.

Sketch of proof

Suppose that Y is a topological space having a complete pair-base and X is a G_δ-subset of Y. Let $\mathcal{P}$ be a complete pair-base for Y and $\{G_n : n \in \mathbb{N}\}$ be a decreasing family of open subsets of Y such that $X = \bigcap_{n\in\mathbb{N}} G_n$. Define the number

$$\delta(V) = \begin{cases} \max\{n \in \mathbb{N} : \overline{V} \subseteq G_n\} & , \quad \overline{V} \nsubseteq G_k \text{ for a } k \in \mathbb{N}, \\ \infty & , \quad \text{otherwise}, \end{cases}$$

for each open subset V of X. The family

$$\mathcal{P}_X = \{(U,V) : U, V \subseteq X \text{ open}, \overline{U}^X \subseteq V, \delta(U) > \delta(V) \text{ if } \delta(V) < \infty, \text{ and } \exists (A,B) \in \mathcal{P}; \overline{U} \subseteq A \subseteq B \subseteq \overline{V}\}$$

is a complete pair-base for X.

Theorem

The union of two spaces having a complete pair-base has a complete pair-base.

Sketch of proof

Let $X = Y \cup Z$, and let $\mathcal{Q}$ and $\mathcal{R}$ be complete pair-bases for Y and Z, respectively. Then the family

$\mathcal{P} = \{(A, B) : A, B \subseteq X$, A is open, $\overline{A} \subseteq B$, $(A \cap Y, B \cap Y) \in Q$ or $(A \cap Z, B \cap Z) \in \mathcal{R}\}$

is a complete pair-base for X.

Here is one more common trait with subcompactness and domain representability.

Theorem

If X is a topological space and $\mathcal{O}$ is a family of open subspaces having a complete pair-base, then $\bigcup \mathcal{O}$ has a complete pair-base.

Sketch of proof

Let $\mathcal{O} = \{O_\alpha : \alpha \in \kappa\}$ where κ is a cardinal number. We can assume that $X = \bigcup_{\alpha \in \kappa} O_\alpha$. Let $\mathcal{P}_\alpha$ be a complete pair-base for O_α, for each $\alpha \in \kappa$. The family

$$\mathcal{P} = \bigcup_{\alpha \in \kappa} \left\{ (A, B) \in \mathcal{P}_\alpha : B \not\subseteq \bigcup_{\beta < \alpha} O_\beta \right\}$$

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Corollary

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We note again that it is an open problem whether Čech-complete spaces are subcompact or not.

MAIN RESULTS

Theorem

p-spaces have an L-complete pair-base.

First, we recall that the internal characterization of *p*-spaces given by Burke.

A space X is **p-space** if and only if there exists a sequence $\{\mathcal{O}_n : n \in \mathbb{N}\}$ of open covers of X satisfying:

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i−) $\bigcap_{n\in\mathbb{N}} \overline{O_n}$ is compact,

ii−) $\{\bigcap_{i\leq n} \overline{O_i} : n \in \mathbb{N}\}$ is an outer network for the set $\bigcap_{n\in\mathbb{N}} \overline{O_n}$.

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So, we can state

Corollary

Let X be a paracompact p-space. If X has a countably complete pair-base then X is Čech-complete.

Proof.

Let $\mathcal{Q}$ be a countably complete pair-base for X. Since X is p-space, X has also an L-complete pair-base $\mathcal{R}$ by the previous theorem. Then the composition of the families $\mathcal{Q}$ and $\mathcal{R}$,

$$\mathcal{P} = \mathcal{Q} \circ \mathcal{R} = \{(A, D) : \exists B, C; (A, B) \in \mathcal{Q}, \ (C, D) \in \mathcal{R} \text{ and } B \subseteq C\}$$

is a complete pair-base for X. Therefore X is domain representable. At the same time, we know that paracompact p-spaces are the perfect pre-images of metric spaces. So, we have a metric space Y and a perfect onto map $f : X \to Y$, and hence Y is domain representable. Since domain representability is equivalent to Čech-completeness in metrizable spaces, we have the space Y is Čech-complete. Hence the space X is Čech-complete by 3.9.10. Theorem in [5] . □

[5] Ryszard Engelking, *General Topology*, Heldermann Verlag, Berlin, 1989.

We are still investigating the maps that leaves having a complete pair-base invariant. At the moment, all we can say is it is "partially" invariant under retractions.

First, let us recall that a **retraction** from the topological space Y onto a subspace X of Y is a continuous map $r : Y \to X$ such that $r(x) = x$ for all $x \in X$, and if that is the case, X is called a **retract** of Y.

First, let us recall that a **retraction** from the topological space Y onto a subspace X of Y is a continuous map $r : Y \to X$ such that $r(x) = x$ for all $x \in X$, and if that is the case, X is called a **retract** of Y.

It is proved that retracts of domain representable spaces are domain representable in

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MAIN RESULTS

retracts

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We could not obtain that the property of having a complete pair-base is preserved under retractions but obtained the following:

Theorem

Every retract of a space having a fip-complete pair-base has a complete pair-base.

Sketch of proof

Let the space Y have a fip-complete pair-base $\mathcal{P}$ and let X be a retract of Y with the retraction $r : Y \to X$. Then the family

$$\mathcal{P}_X = \{(A \cap X, r(B)) : (A, B) \in \mathcal{P}\}$$

is a complete quasi-pair-base for X. So, X has a complete pair-base.

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T H A N K Y O U