6-2017

Entropy of Induced Continuum Dendrite Homeomorphisms

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Topological entropy of induced continuum dendrite homeomorphisms

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June-2017

Scheme

1. Introduction
2. Problem
3. Sketch of the proof
4. Remark

Given a compact metric space X,

$$2^X = \{A \subset X : A \text{ is a nonempty and closed in } X\} \rightarrow \textit{Hyperspace}$$

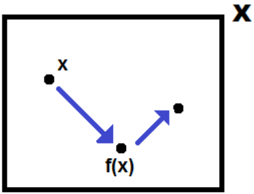
If X is a continuum

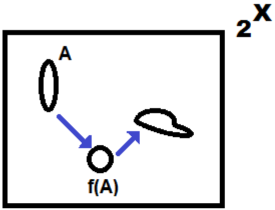
$$C(X) = \{A \in 2^X : A \text{ is connected in } X\} \rightarrow \textit{Continuum hyperspace}$$

Induced map

If $f : X \rightarrow X$ is a continuous map on a compact metric space then its induced map $2^f : 2^X \rightarrow 2^X$ is defined as $2^f(A) = f(A)$.
When X is a continuum space then its **induced continuum map** is defined as $C(f) = 2^f|_{C(X)}$.

Introduction





Theorem (Bauer-Sigmund(1975))

Let $f : X \to X$ be a continuous map with $h(f) > 0$ then $h(2^f) = \infty$.

- $0 < h(f) = h(2^f|_{K_1(X)}) \leq h(C(f))$.
- The tent map does not hold Theorem. Besides if f is the tent map then $h(f) = h(C(f)) = \log 2$.
- What happens with the topological entropy of the induced maps 2^f and $C(f)$ if the topological entropy of f is zero?

Introduction

Theorem (Lampart-Raith(2010))

Let $f : I \to I$ be a homeomorphism on the interval $I = [0,1]$ then

- *the topological entropy of $2^f : 2^I \to 2^I$ has only two possible values: 0 or ∞*
- *the topologicla entropy of $2^f : 2^I \to 2^I$ is ∞ if and only if $f \circ f$ is not the identity map.*

Theorem (Lampart-Raith(2010))

Let $f : D \to D$ be a dendrite homeomorphism then the topological entropy of its induced continuum map $C(f)$ is 0.

- D is a dendrite if is a locally connected continuum that contains no simple closed curves.
- $f : D \to D$ is a dendrite homeomorphism or DH if f is a homeomorphism and D is a dendrite.
- The topological entropy of all dendrite homeomorphism is zero.

Theorem (Hernández-Méndez(2016))

Let $f : D \to D$ be a DH then

- *the topological entropy of $2^f : 2^D \to 2^D$ has only two possible values: 0 or ∞*
- *the topological entropy of $2^f : 2^D \to 2^D$ is ∞ if and only if the set of recurrent points of f is distinct from D.*

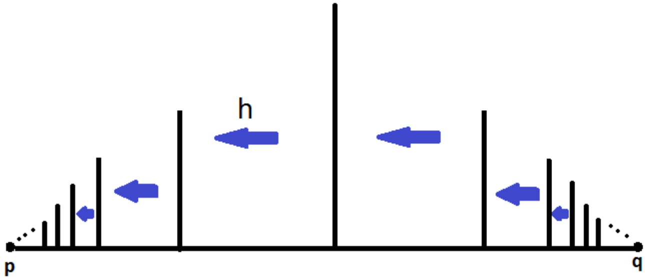
Problem

Question

What happens with the induced continuum dendrite homeomorphism?

There are examples of dendrite homeomorphisms such that the topological entropy of its induced continuum map is ∞.

Example:



Problem

We observed that there are dynamical systems with this dendrite homeomorphism:

- North Pole-South Pole diffeomorphism on S^2.
- The diffeomorphism time-one map defined on the torus of $X = gradh$ where h is the height function of points of the torus above the horizontal plane.

Therefore we introduce the following definition:

Definition

We say that a closed subset $\Lambda \subset X$ is a **Special dendrite** if there is $k \in \mathbb{N}$ such that Λ is f^k-invariant and $f^k|_\Lambda$ is conjugated to h. In this case, we say that f admit a special dendrite.

Problem

Main Theorem

Let $f : D \to D$ be a DH. Then

i) If there is a non-recurrent branch point then $h(C(f)) = \infty$

ii) If every point is a recurrent point then $h(C(f)) = 0$.

- $x \in D$ is an **end point** of D provided that $D \setminus \{x\}$ is connected;
- $x \in D$ is a **cut point** of D if $D \setminus \{x\}$ is not connected;
- $ord(x) =$ is the cardinality of the set of all components od $D \setminus \{x\}$;
- If $ord(x) \geq 3$ then x is a **branch point** of D.

Sketch of the proof

The item *ii*. is a direct consequence of

Theorem (Hernández-Méndez(2015))

If $f : D \to D$ is a DH such that $R(f) = D$. Then $h(2^f) = 0$.

The item *i*. we divided in two cases:

Theorem (A)

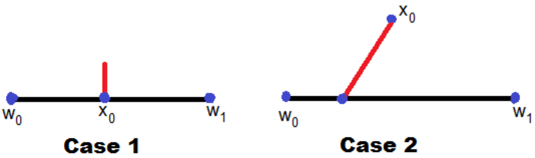
Let $x_0 \in D \setminus R(f)$ such that $ord(x_0) \geq 3$ and U the connected component of $D \setminus R(f)$ that contains x_0. If U is f^n-invariant for some $n \in \mathbb{N}$ then $h(C(f)) = \infty$.

Theorem (B)

Let $x_0 \in D \setminus R(f)$ such that $ord(x_0) \geq 3$ and U the connected component of $D \setminus R(f)$ that contains x_0. If U is not f^n-invariant for all $n \in \mathbb{N}$ then $h(C(f)) = \infty$.

Proof of Theorem A

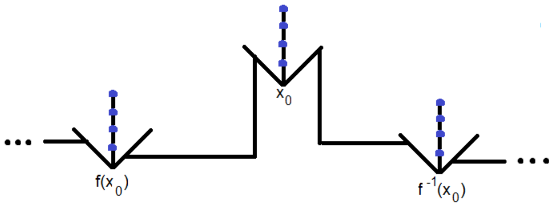
- Let $g = f^N : W \to W$ where $W = \overline{U}$.
- g has only two fixed points w_0 and w_1 (and are end points of W).
- The arc $[w_0, w_1]$ is strongly g-invariant.



Proof of Theorem B

Lemma

Let $k \in \mathbb{N}$, there is $\delta > 0$ such that $s(n+1, C(f)^{-1}, \delta) \geq k^n$ for all $n \in \mathbb{N}$.



Let $f : X \to X$ be a continuous map and X compact metric space

- $E \subset X$ is (n, ε)-separated set if $d_n(x, y) > \varepsilon$ for $x, y \in E$.
- $s(n, \varepsilon)$ = maximal cardinality of (n, ε)-separated set.
- $h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon) \to$ **topological entropy**.

Remark

Using the same technique we show that

Theorem

The induced continuum map of every Morse-Smale diffeomorphism has infinite topological entropy.

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