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On Continua with Regular Non-abelian Self Covers

Mathew Timm
Bradley University, mtimm@fsmail.bradley.edu

Spaces with Regular Nonabelian Self Covers

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Recall: A *covering space* of a topological space Y is a pair (X, f), X a topological space, f continuous, such that $f : X \to Y$ has the property that for each $y \in Y$ there is a neighborhood U_y of y such that $f^{-1}(U_y)$ is a pair wise disjoint collection of open subsets of X, each of which f maps homeomorphically onto U_y.

A deck transformation for the covering space $f : X \to Y$ is a homeomorphism $\alpha : X \to X$ such that $f \circ \alpha = f$. $Aut_Y(X, f)$ denotes the collection of all deck transformations for $f : X \to Y$. It is a group under function composition. $f : X \to Y$ is a *regular covering*, if for each $y \in Y$ and every pair $x_1, x_2 \in f^{-1}(y)$ there is an $\alpha \in Aut_Y(X, f)$ such that $\alpha(x_1) = x_2$.

Equivalently, for nice spaces, $f : X \to Y$ is a regular cover if $f_*(\pi_1(X, x_0))$ is a normal subgroup of $\pi_1(Y, y_0)$.

Recall that for nice spaces, when $f : X \to Y$ is a regular k-to-1 covering space, then $|\pi_1(Y)/f_*(\pi_1(X))| = k$ and $Aut_Y(X, f) \cong \pi_1(Y)/f_*(\pi_1(X))$

Finally, when the finite group G, $|G| \geq 2$, is the group of deck transformations for the regular cover $f : X \to Y$ we say f is a G *regular cover of* Y and when $f : Y \to Y$ is a G regular cover of Y with $|G| \geq 2$, we say f *is a* G *regular self cover*.

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3. Let $\boldsymbol{p} = (p_i)_{i=1}^{\infty}$ be a sequence of primes. Then the solenoid $\Sigma_{\boldsymbol{p}}$ has $\mathbb{Z}_q$ regular self covers for each prime q which does not appear infinitely often in $\boldsymbol{p}$.

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4. The pseudo circle has a $\mathbb{Z}_n$ regular self covers for all n.
   $\mathbb{Z}_2$ case -- Jo Heath, $\mathbb{Z}_n$ case -- David Bellamy.

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[1] To appear Houston J. Math.

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Question. Are their continua which have G regular self covers for some nonabelian group G and which have less complicated local topology?

Theorem. There is a disk with holes K and a Cantor set $C \subset K$ such that K has G regular self covers for every finite group G and such that $K \setminus C$ is a planar 2-dim manifold with boundary.

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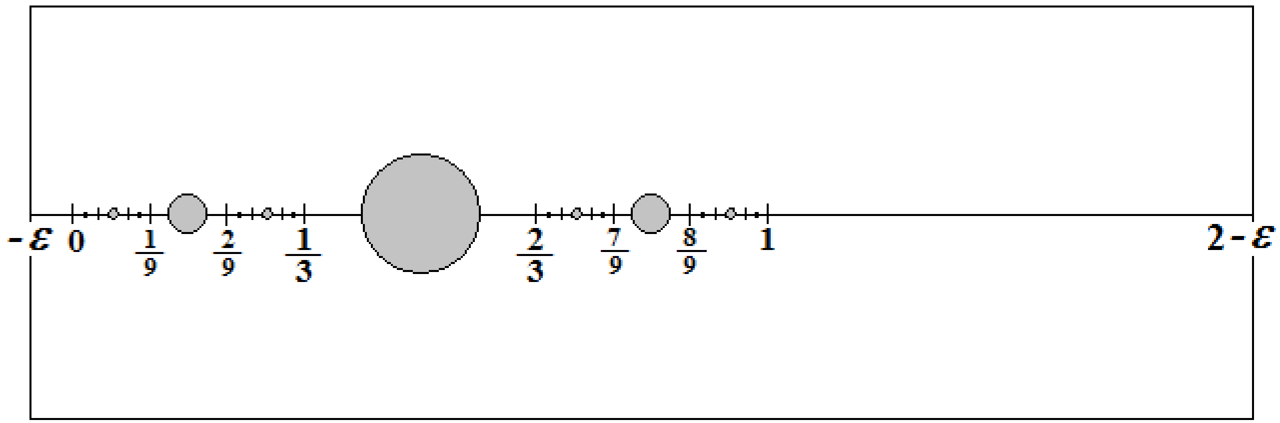

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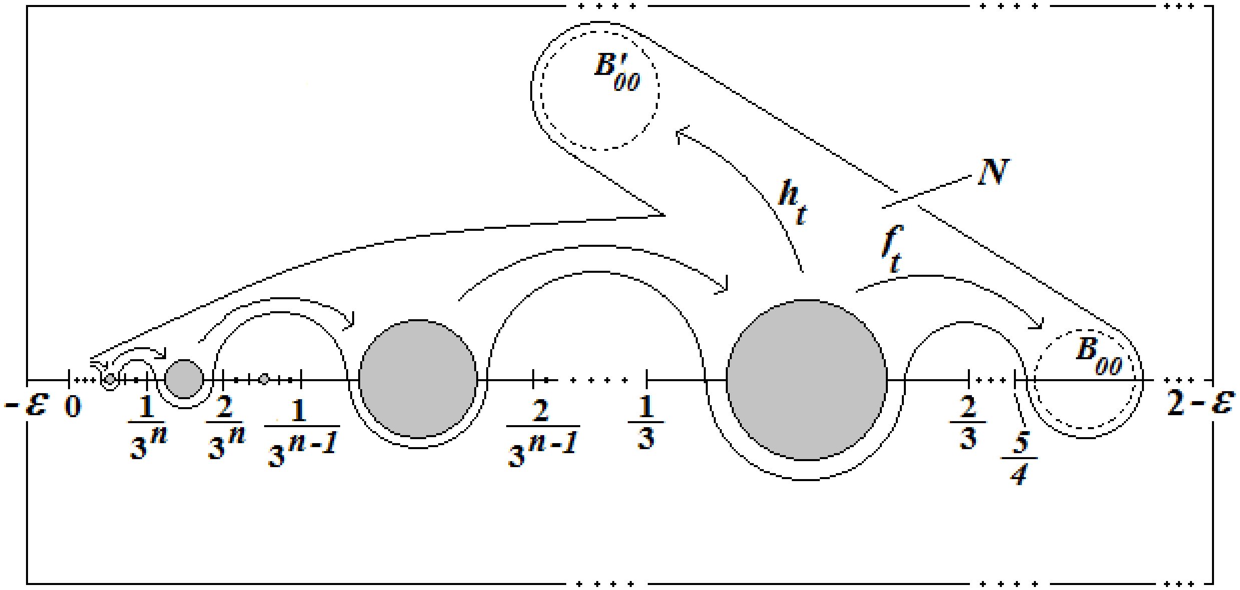
Proof:



The Method of Disks with Holes.

1. Given any finite group G, there is a G regular cover $\theta = \theta_G : X \to M$ of a disk M with some number m holes by a disk X with some number n of holes.

2. M and X can be assumed to be coordinatized so that each is a copy of the 2-disk $[-\epsilon, 2-\epsilon] \times [-1, 1]$ with holes punched into it.

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- Without loss of generality we assume
  - $C \subset [0, 1] \times 0 \subset M$
  - $C' \subset [0, 1] \times 0 \subset X$

Punch countably many holes of small radius into M inside of E centers at the center of the complementary intervals used to form C and use θ^{-1} to lift the holes to X. Note, the holes in X are between the rational points of C'.

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This turns M into a copy of K_M of K with what appear to be m extra holes and X into a copy of K_X of K with appearently n extra holes.

But, by the Hole Isotopy Lemma, these extra holes can be isotoped into positions "between" point$s$ in C and C'. Thus K_M and K_X are copies of K and the restriction

$$\theta : K = K_X \to K = K_M$$

is a G regular self cover of K. $\square$

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1. (Daverman) Does there exist a finite non-abelian group G and a compact connected n manifold M^n $(n = n(G))$ such that M^n has a regular G self cover $\theta : M^n \to M^n$?

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1. (Daverman) Does there exist a finite non-abelian group G and a compact connected n manifold M^n $(n = n(G))$ such that M^n has a regular G self cover $\theta : M^n \to M^n$?

2. Let $\mathcal{F} = \{f_n : n \in \mathbb{N}\}$ be a countably infinite collection of pairwise "inequivalent" G_n regular self covers, at least some G_n nonabelian, of the metric continuum X. Must there be a Cantor set $C \subset X$ around which X has bad local topology? If not, can you characterize when this happens?

Thank you.