6-2017

The Isbell-hull of an Asymmetrically Normed Space

Olivier Olela Otafudu
North-West University (South Africa), olivier.olelaotafudu@nwu.ac.za

Jurie Conradie

Hans-Peter Künzi

The Isbell-hull of an asymmetrically normed space

Olivier Olela Otafudu
School of Mathematical Sciences
North-West University (Mafikeng Campus)

University of Dayton, Dayton, Ohio, USA

Asymmetrically normed spaces

A function $p : X \to [0, \infty)$ on a real vector space X will be called an asymmetric seminorm on X if for all $x, y \in X$ and $\lambda \in [0, \infty)$,
(a) $p(\lambda x) = \lambda p(x)$;
(b) $p(x + y) \leq p(x) + p(y)$.
If in addition we also have
(c) $p(x) = p(-x) = 0$ if and only if $x = 0$,
p will be called an asymmetric norm, and the pair (X, p) an asymmetrically normed space.
If (a) is replaced by by:
(a') $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$, then p is called a semi-norm.
If p is an asymmetric norm on X, the function $p^t : X \to [0, \infty)$ defined by

$$p^t(x) = p(-x), \;\; x \in X$$

is also an asymmetric norm, the asymmetric norm conjugate to p.
The symmetrisation of the asymmetric norm p is the function $p^s : X \to [0, \infty)$ given by

$$p^s(x) = \max\{p(x), p(-x)\}, \;\; x \in X$$

and this is easily seen to be a norm on X.

An asymmetric norm p induces a T_0-quasi-metric d_p on X defined by

$$d_p(x, y) = p(y - x) \text{ for all } x, y \in X.$$

For $x \in X, r > 0$ we define the balls

$$B_r^p(x) = \{y \in X : d_p(x, y) < r\} = \{y \in X : p(y - x) < r\}$$

and

$$B_r^p[x] = \{y \in X : d_p(x, y) \leq r\} = \{y \in X : p(y - x) \leq r\}.$$

The family $\{B_r^p(x) : r > 0]\}$ forms a fundamental system of neighbourhoods for x for a T_0 topology τ_p on X, which we shall refer to as the topology induced by p.
The Hausdorff topology τ_{p^s} induced by the norm p^s is clearly finer than the topologies τ_p and τ_{p^t}.

Example

As a simple but important example we mention the asymmetric norm p_1 on $\mathbb{R}$ (regarded as a real vector space) defined for all $x \in \mathbb{R}$ by

$$p_1(x) = x^+,$$

where $x^+ = x \vee 0 = \max\{x, 0\}$ is the positive part of x. In this case

$$p_1^t(x) = \max\{-x, 0\} = x^-$$

$$p_1^s(x) = \max\{x^+, x^-\} = |x|.$$

Definition

An asymmetrically normed space (X, p) will be called (a) Isbell-convex if for every family $(x_i)_{i\in I}$ of elements of X and families of non-negative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ it follows that if

$$p(x_j - x_i) \leq r_i + s_j$$

whenever $i, j \in I$, then

$$\bigcap_{i\in I} B^p_{r_i}[x_i] \cap B^{p^t}_{s_i}[x_i] \neq \emptyset.$$

(b) metrically convex if for every two elements $x, y \in X$ and non-negative numbers r and s such that $p(y - x) \leq r + s$, there exists a $z \in X$ such that $p(z - x) \leq r$ and $p(y - z) \leq s$.

(c) Isbell-complete if for each family $(x_i)_{i\in I}$ of elements in X and families of non-negative real numbers $(r_i)_{i\in I}$ and $(s_i)_{i\in I}$ such that if $B^p_{r_i}[x_i] \cap B^{p^t}_{s_j}[x_j] \neq \emptyset$ whenever $i, j \in I$, then

$$\bigcap_{i\in I} B^p_{r_i}[x_i] \cap B^{p^t}_{s_i}[x_i] \neq \emptyset.$$

Examples

1. $X = \mathbb{R}^2$, $\|(x_1, x_2)\| = \max\{|X_1|, |X_2|\}$ has the binary intersection property.
2. $X = \mathbb{R}^2$, $\|(x_1, x_2)\| = \sqrt{x_1^2 + x_2^2}$ does not have the binary intersection property.
3. Hence $\mathbb{C}$, with its usual norm, does not have the binary intersection property.

Note that an asymmetrically normed space (X, p) is Isbell-convex (metrically convex, Isbell-complete) if and only if the T_0-quasi-metric space (X, d_p) has the same property.

Note that an asymmetrically normed space (X, p) is Isbell-convex (metrically convex, Isbell-complete) if and only if the T_0-quasi-metric space (X, d_p) has the same property.

Lemma

Every asymmetrically normed space (X, p) is metrically con- vex.

Note that an asymmetrically normed space (X, p) is Isbell-convex (metrically convex, Isbell-complete) if and only if the T_0-quasi-metric space (X, d_p) has the same property.

Lemma

Every asymmetrically normed space (X, p) is metrically con- vex.

Lemma

An asymmetrically normed space (X, p) is Isbell-convex if and only if it is Isbell-complete.

Definition

An asymmetrically normed space (Y, q) is called 1-injective if for every asymmetrically normed space (X, p) and every linear subspace X_0 of X, every continuous linear map $T_0 : (X_0, p) \to (Y, q)$ has a continuous extension $T : X \to Y$ such that $\|T\|_{p,q} \leq \|T_0\|_{p,q}$.

Lemma

If the asymmetrically normed space (X, p) is Isbell-convex, then so is (X, p^t), and the normed space (X, p^s) is a hyperconvex Banach space, and therefore 1-injective (as a Banach space).

Definition

An asymmetrically normed space (Y, q) is called 1-injective if for every asymmetrically normed space (X, p) and every linear subspace X_0 of X, every continuous linear map $T_0 : (X_0, p) \to (Y, q)$ has a continuous extension $T : X \to Y$ such that $\|T\|_{p,q} \leq \|T_0\|_{p,q}$.

Lemma

If the asymmetrically normed space (X, p) is Isbell-convex, then so is (X, p^t), and the normed space (X, p^s) is a hyperconvex Banach space, and therefore 1-injective (as a Banach space).

Theorem

An Isbell-convex (equivalently, Isbell-complete) asymmetrically normed space (Y, q) is 1-injective.

Let (X, p) be an asymmetrically normed space. Recall that a function pair $f = (f_1, f_2)$, where $f_i : X \to [0, \infty)$ for $i = 1, 2$, is called ample if

$$p(y - x) \leq f_2(x) + f_1(y),$$

and that f is minimal whenever $g = (g_1, g_2)$ is an ample pair such that if

$$g_1(x) \leq f_1(x),$$

$$g_2(x) \leq f_2(x)$$

for all $x \in X$, then $g_1 = f_1, g_2 = f_2$. The set of all minimal function pairs on X will be denoted by $\mathcal{E}(X, p)$. The following characterisation of the elements of $\mathcal{E}(X, p)$ is useful; recall that for $a \in \mathbb{R}$, we write $a^+ = a \vee 0 = \max\{a, 0\}$.

Let (X, p) be an asymmetrically normed space. Recall that a function pair $f = (f_1, f_2)$, where $f_i : X \to [0, \infty)$ for $i = 1, 2$, is called ample if

$$p(y - x) \leq f_2(x) + f_1(y),$$

and that f is minimal whenever $g = (g_1, g_2)$ is an ample pair such that if

$$g_1(x) \leq f_1(x),$$

$$g_2(x) \leq f_2(x)$$

for all $x \in X$, then $g_1 = f_1, g_2 = f_2$. The set of all minimal function pairs on X will be denoted by $\mathcal{E}(X, p)$. The following characterisation of the elements of $\mathcal{E}(X, p)$ is useful; recall that for $a \in \mathbb{R}$, we write $a^+ = a \vee 0 = \max\{a, 0\}$. A function pair $f = (f_1, f_2)$ belongs to $\mathcal{E}(X, p)$ if and only if for every $x \in X$,

$$f_2(x) = \sup_{s \in X} (p(s - x) - f_1(s))^+$$

and

$$f_1(x) = \sup_{s \in X} (p(x - s) - f_2(s))^+.$$

For every $z \in X$, we define the minimal function pair $f_z = (f_{z,1}, f_{z,2})$ by

$$f_{z,1}(x) = p(x - z), \qquad f_{z,2}(x) = p(z - x).$$

The mapping $z \mapsto f_z$ is an injection of X into $\mathcal{E}(X, p)$.

For every $z \in X$, we define the minimal function pair $f_z = (f_{z,1}, f_{z,2})$ by

$$f_{z,1}(x) = p(x - z), \qquad f_{z,2}(x) = p(z - x).$$

The mapping $z \mapsto f_z$ is an injection of X into $\mathcal{E}(X, p)$.
We now define scalar multiplication on $\mathcal{E}(X, p)$. For $\lambda \in \mathbb{R}$ and $f \in \mathcal{E}(X, p)$, we define the function pair $f^\lambda = (f_1^\lambda, f_2^\lambda)$ by

$$f_1^\lambda(x) = \begin{cases} \lambda f_1(\lambda^{-1}x) & \text{if } \lambda > 0, \\ p(x) & \text{if } \lambda = 0, \\ |\lambda| f_2(\lambda^{-1}x) & \text{if } \lambda < 0 \end{cases} \quad \text{and}$$

$$f_2^\lambda(x) = \begin{cases} \lambda f_2(\lambda^{-1}x) & \text{if } \lambda > 0, \\ p(-x) & \text{if } \lambda = 0, \\ |\lambda| f_1(\lambda^{-1}x) & \text{if } \lambda < 0. \end{cases}$$

Lemma

If $f = (f_1, f_2) \in \mathcal{E}(X, p)$ and $\lambda \in \mathbb{R}$, then $f^\lambda \in \mathcal{E}(X, p)$

It now follows that we can define scalar multiplication in $\mathcal{E}(X, p)$ by putting

$$\lambda f = f^\lambda.$$

We now turn to defining addition on $\mathcal{E}(X, p)$. If $f = (f_1, f_2), g = (g_1, g_2) \in \mathcal{E}(X, p)$, $x \in X$ we put $f \oplus g = ((f \oplus g)_1, (f \oplus g)_2)$, where

$$(f \oplus g)_1(x) = \sup\{(f_1(x - s) - g_2(s))^+ : s \in X\},$$
$$(f \oplus g)_2(x) = \sup\{(f_2(x - s) - g_1(s))^+ : s \in X\}.$$

Lemma (Olela Otafudu, Topology Appl. 166 (2014))

If $f = (f_1, f_2) \in \mathcal{E}(X, p)$, then for $x \in X$,

$$\sup\{(f_1(x + s) - f_1(s))^+ : s \in X\} = p(x),$$

$$\sup\{(f_2(x + s) - f_2(s))^+ : s \in X\} = p(-x).$$

If $f, g \in \mathcal{E}(X, p)$, then $f \oplus g$ is ample.

If $f, g \in \mathcal{E}(X, p)$, then $f \oplus g$ is ample.
Suppose $x, y, z \in X$. Then

$$(f_y \oplus f_z)_1(x) = f_{(y+z),1}(x)$$

and

$$(f_y \oplus f_z)_2(x) = f_{(y+z),2}(x).$$

Lemma

If $f, g \in \mathcal{E}(X, p)$ and $x \in X$, then

$$\sup_{s \in X}(f_1(x-s) - g_2(s))^+ = \sup_{s \in X}(g_1(s) - f_2(x-s))^+$$

and

$$\sup_{s \in X}(f_2(x-s) - g_1(s))^+ = \sup_{s \in X}(g_2(s) - f_1(x-s))^+.$$

Let $f, g, h \in \mathcal{E}(X, p)$.
Then

$$f \oplus g = g \oplus f$$

and
suppose $f \oplus g, g \oplus h \in \mathcal{E}(X, p)$. Then

$$(f \oplus g) \oplus h = f \oplus (g \oplus h).$$

Let $f, g, h \in \mathcal{E}(X, p)$.
Then

$$f \oplus g = g \oplus f$$

and
suppose $f \oplus g, g \oplus h \in \mathcal{E}(X, p)$. Then

$$(f \oplus g) \oplus h = f \oplus (g \oplus h).$$

In the light of the definition of scalar multiplication, the only candidate for the additive identity is $f^0 = (f_1^0, f_2^0)$, with $f_1^0(x) = p(x), f_2^0(x) = p(-x))$. We check this: Let $f = (f_1, f_2) \in \mathcal{E}(X, p)$ and $x \in X$. Then,

$$(f^0 \oplus f)_1(x) = \sup\{(p(x - s) - f_2(s))^+ : s \in X\} = f_1(x),$$

and

$$(f^0 \oplus f)_2(x) = \sup\{(p(s - x) - f_1(x))^+ : s \in X\} = f_2(x).$$

The only candidate for the additive inverse of $f = (f_1, f_2)$ is $(-1)f = (f_1^{-1}, f_2^{-1})$. We check this:

$$(f \oplus f^{-1})_1(x) = \sup\{(f_1(x-s) - f_2^{-1}(s))^+ : s \in X\}$$
$$= \sup\{(f_1(x+s) - f_1(s))^+ : s \in X\}$$
$$= p(x) = f_1^0(x).$$

A similar calculation shows that

$$(f \oplus f^{-1})_2(x) = p^{-1}(x) = f_2^0(x).$$

We denote the additive inverse of $f = (f_1, f_2) \in \mathcal{E}(X, p)$ by $-f$; thus $-f = ((-f)_1, (-f)_2)$, where

$$(-f)_1(x) = f_2(-x),$$

$$(-f)_2(x) = f_1(-x)$$

whenever $x \in X$. If $f, g \in \mathcal{E}(X, p)$, then $f \oplus g \in \mathcal{E}(X, p)$.

Theorem

If scalar multiplication on $\mathcal{E}(X,p)$ is defined by $\lambda f = f^{\lambda}$ and addition $\oplus$ by

$$(f \oplus g)_1(x) = \sup\{(f_1(x-s) - g_2(s))^+ : s \in X\}$$

and

$$(f \oplus g)_2(x) = \sup\{(f_2(x-s) - g_1(s))^+ : s \in X\}$$

whenever $f, g \in \mathcal{E}(X,p)$ and $\lambda \in \mathbb{R}$, then $\mathcal{E}(X,p)$ is a vector space and the map $x \mapsto f_x$ is a linear isomorphism of X into $\mathcal{E}(X,p)$.

To define an asymmetric norm on $\mathcal{E}(X,p)$ we take our cue from the T_0-quasi-metric D defined on the injective hull of a T_0-quasi-metric space (X,d) in [Kemajou, Künzi, Otafudu, Topology Apppl. 159 (2012)] by

$$D(f,g) = \max\{\sup_{s\in X}(f_1(s) - g_1(s))^+, \sup_{s\in X}(g_2(s) - f_2(s))^+\}$$

for $f,g \in \mathcal{E}(X,p)$, it is shown that

$$D(f,g) = \sup_{s\in X}(f_1(s) - g_1(s))^+ = \sup_{s\in X}(g_2(s) - f_2(s))^+.$$

Recall that the additive identity $f^0 = (f_1^0, f_2^0)$ is defined by

$$f_1^0(s) = p(s), f_2^0(s) = p(-s).$$

For $f \in \mathcal{E}(X,p)$ we now put

$$\widetilde{p}(f) = D(f^0, f) = \sup_{s\in X}(f_2(s) - p(-s))^+ = \sup_{s\in X}(p(x) - f_1(s))^+ = f_2(0)$$

and

$$\widetilde{p}(-f) = D(f, f^0) = f_1(0).$$

The function $\widetilde{p} : \mathcal{E}(X, p) \to [0, \infty)$ defined above is an asymmetric norm on $\mathcal{E}(X, p)$ and the map $x \mapsto f_x$ is an isometry.

The function $\widetilde{p} : \mathcal{E}(X, p) \to [0, \infty)$ defined above is an asymmetric norm on $\mathcal{E}(X, p)$ and the map $x \mapsto f_x$ is an isometry.

Theorem

An 1-injective asymmetrically normed space (X, p) is Isbell-convex.

The function $\widetilde{p} : \mathcal{E}(X, p) \to [0, \infty)$ defined above is an asymmetric norm on $\mathcal{E}(X, p)$ and the map $x \mapsto f_x$ is an isometry.

Theorem

An 1-injective asymmetrically normed space (X, p) is Isbell-convex.

Lemma

$\mathcal{E}(X, p)$ *is Isbell-convex.*

- E. Kemajou, H.-P. Künzi and O. Olela Otafudu, The Isbell-hull of a di-space, Topology Appl. 159 (2012) 2463–2475.
- J.R. Isbell, Six theorems about injective metric spaces, Comment. Math. Helv. 39 (1964), 65–76.
- O. Olela Otafudu, Extremal function pairs in asymmetric normed linear spaces, Topology Appl. 166 (2014) 98-107.