Numerical Investigation into a Computational Approximation of Bifurcation Curves

Josh Craven
Advisor: Dr. Muhammad Usman
Department of Mathematics, University of Dayton

Abstract
In this project, we use computational tools to study bifurcations in nonlinear oscillations. MATLAB is first used to determine the slow flow phase portrait of each region and the characteristics of each critical point. Next, we parameters are disrupted and for each set of values we find the locations of the real critical points and the eigenvalues of the Jacobian matrix. With this knowledge, we can approximate the bifurcation diagram. These results are compared with results from previous software.

Background
Finding exact solutions to differential equations such as the forced Mathieu-van der Pol equation can be very difficult, but there are ways of approximating the solution. Numerical integration is one such way of approximating the solution. Once an approximate solution is obtained, we would also like to know how the characteristics of the in the equation affects the answer. We do this by finding the slow flow equations and then see how changing the parameters affects the characteristics of the slow flow phase portrait. A bifurcation diagram shows the different types of slow flow for given parameter values. If two slow flow phase portraits have the same number of each type of discontinuity, then they are said to be in the same region. We will develop a method of finding and classifying the discontinuities of the slow flow for given parameter values. With this method we will be able to generate bifurcation diagrams for problems that have two slow flow equations.

Using perturbation methods, we can obtain equations for the slow flow. The phase portrait of the slow flow is obtained by picking different starting points, simultaneously approximating the slow flow equations at each point, and then plotting all of the resulting paths together. Discontinuities will appear in the plot where sets of lines start or end or axis. Points with path starting and moving away from them are known as sources while points with lines ending and moving towards them are known as sinks. Some points have lines moving towards them along an axis and lines moving away from them along a perpendicular axis. These types of points are known as saddle. Path lines may even converge or diverge from a point. The point inside the ring that the lines converge to or diverge from is the critical point and is known as a limit cycle.

Finding Eigenvalues
As mentioned above, we will determine the characteristics of the slow flow by finding and classifying each discontinuity. Also, we will only be looking at problems that have two slow flow equations. We will demonstrate this process by applying it to the Mathieu-van der Pol Equation. This equation is given by:

\[ x'' + (1 + \alpha \cos(2t))x - x^3 = \alpha \cos(t) \]

And the slow flow equations are:

\[ \begin{align*}
A' &= -A - x \\
B' &= -B - x
\end{align*} \]

Where \( A \) and \( B \) are functions of \( t \), and \( \alpha, \omega, \) and \( F \) are parameters. We set:

\[ \begin{align*}
A &= \cos(t) \\
B &= \cos(2t)
\end{align*} \]

A critical point occurs wherever \( P(A,B) = 0 \). We call this critical point \( (A,B) \). Mathematica's solve function was used to find any real critical points for \( F = 0 \). After that we find the Jacobian matrix \( J \) for linearization which is:

\[ J = \begin{bmatrix}
\frac{\partial A}{\partial A} & \frac{\partial A}{\partial B} \\
\frac{\partial B}{\partial A} & \frac{\partial B}{\partial B}
\end{bmatrix} \]

The next step is to find the eigenvalues, \( \lambda \), of the Jacobian for each set of critical points. There should be two eigenvalues, and we finish by solving the following equation for \( \lambda \):

\[ |J - \lambda I| = 0 \]

\( I \) represents the 2x2 identity matrix and \( \lambda \) is the determinant. Solving for \( \lambda \) gives us:

\[ \lambda = \frac{-A \pm \sqrt{A^2 + 4B}}{2} \]

The eigenvalues determine which type of discontinuity occurs at each critical point. If the two eigenvalues are real and positive, the point is a source. If they are real and negative, then the point is a sink. Complex, but not purely imaginary eigenvalues indicate a spiral and purely imaginary eigenvalues indicate a center. The following table summarizes the relationship between the eigenvalues and the type of discontinuity:

**Types of Slow Flow by Region**

<table>
<thead>
<tr>
<th>Region</th>
<th>Region sketch</th>
<th>Plot of region</th>
<th>discontinuities</th>
<th>node</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td></td>
<td>2 sinks</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td></td>
<td></td>
<td>2 saddles</td>
<td>[2,2,1,0]</td>
</tr>
<tr>
<td>C1</td>
<td></td>
<td></td>
<td>1 source</td>
<td></td>
</tr>
<tr>
<td>C2</td>
<td></td>
<td></td>
<td>0 spirals</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td></td>
<td></td>
<td>1 saddle</td>
<td>[1,1,0]</td>
</tr>
<tr>
<td>E</td>
<td></td>
<td></td>
<td>0 sinks</td>
<td>[0,0,1,1]</td>
</tr>
<tr>
<td>F</td>
<td></td>
<td></td>
<td>1 saddle</td>
<td>[1,1,0]</td>
</tr>
<tr>
<td>G</td>
<td></td>
<td></td>
<td>1 source</td>
<td></td>
</tr>
<tr>
<td>H</td>
<td></td>
<td></td>
<td>0 spirals</td>
<td></td>
</tr>
<tr>
<td>I</td>
<td></td>
<td></td>
<td>1 stable cycle</td>
<td></td>
</tr>
</tbody>
</table>

Note that regions B and E are essentially the same thing.

Varying Constants
We want to know how varying \( \alpha, k_1, \) and \( F \) will affect the quantity and type of each discontinuity in the slow flow. Note that \( P(A,B) \) and \( Q(A,B) \) remain unchanged when \( A, B, \) and \( F \) are replaced by \( A, B, \) and \( F \), so we can consider \( \Gamma \) to be without loss of generality. The same applies to \( A, k_1, \) and \( \alpha \) so we may also consider \( \alpha = 0 \) wlog. The constants are varied like:

\[ \begin{align*}
0 \leq F \leq 1.5 \\
-75 \leq k_1 \leq 75
\end{align*} \]

I then incremented \( F \) by \( k_1 \) over these ranges and checked which type of slow flow occurred at each set of points. This data is shown in the "Bifurcation Plots" section where each plot shows \( k_1 \) vs. \( F \) for a specific value of \( \alpha \).

Bifurcation Plots
Here we show the approximate bifurcation curve for \( k_1 \) vs. \( F \) for \( \alpha = 1, 2, \) and 0. The black and white plots were generated using another bifurcation software called AUTO. This is shown as a comparison to the colored plots that \( \Gamma \) generated. These plots are very close. At \( \alpha = 1 \), there appears to be a problem at \( k_1 = 25 \). This corresponds to there being no constant term in the square root of the eigenvalue. This would require a more refined perturbation approximation. Similar refined approximations are needed in region A of Figure 2, and region B of Figure 3.

Conclusion
Using perturbation methods and basic bifurcation theory, we were able to find the slow flow characteristics in each region and approximate the bifurcation curves for \( k_1, \) and \( F \). MATLAB was used to generate slow flow and bifurcation curve plots. Certain parts of the bifurcation plots did not match the results of others, especially when \( \alpha = 0 \), so more precise perturbation approximations are required. Overall, these bifurcation curves give a good indication of the slow flow variation as parameters change.

References and Acknowledgements