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Lifting Homeomorphisms of Cyclic Branched Covers of the Sphere

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Lifting homeomorphisms of cyclic branched covers of the sphere

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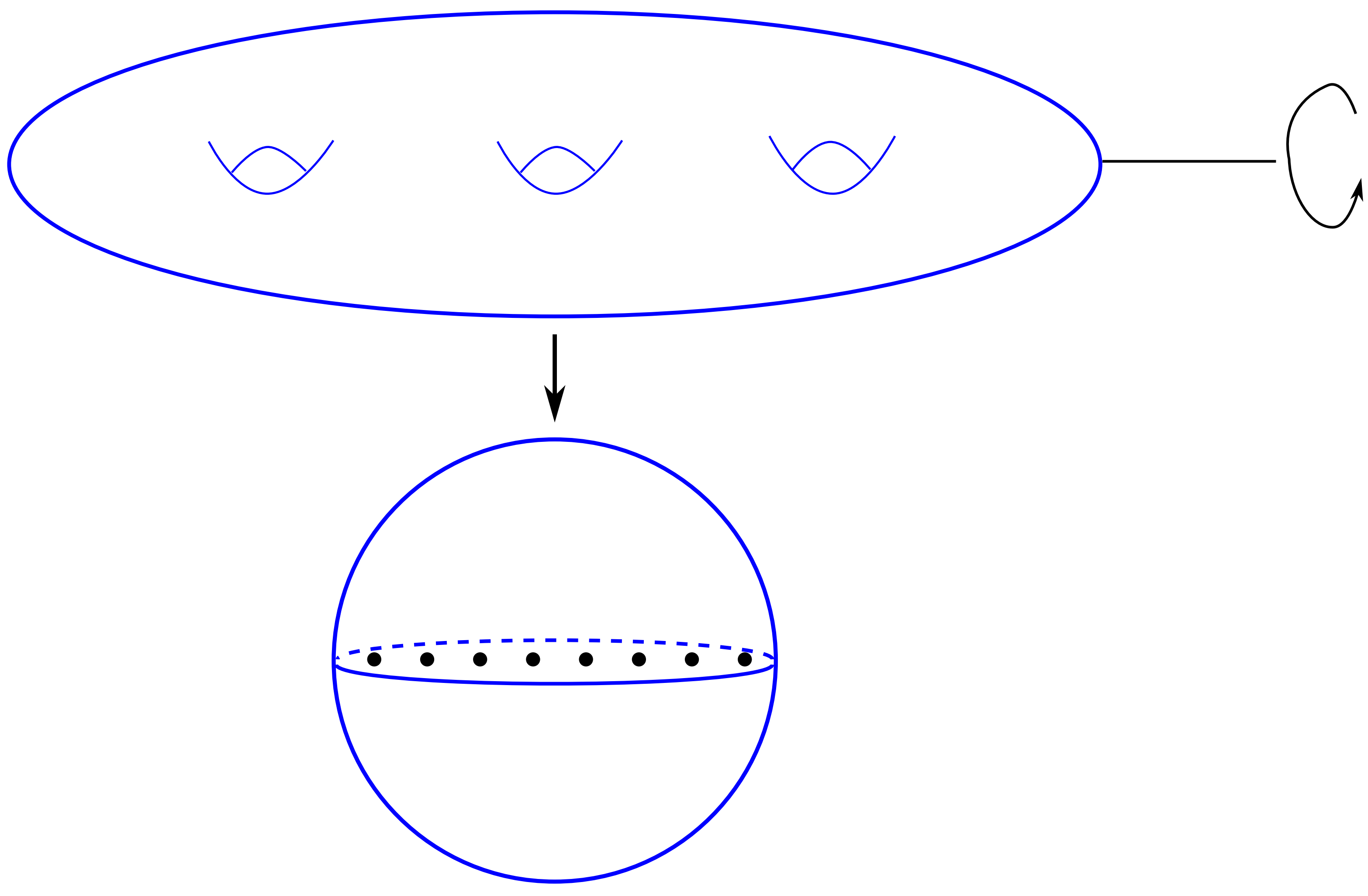
Cyclic Branched Covers of the Sphere

$p : S_g \to S^2$: cyclic branched cover of a sphere.



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Question (Birman—Hilden)

When do all homeomorphisms of S^2 lift?

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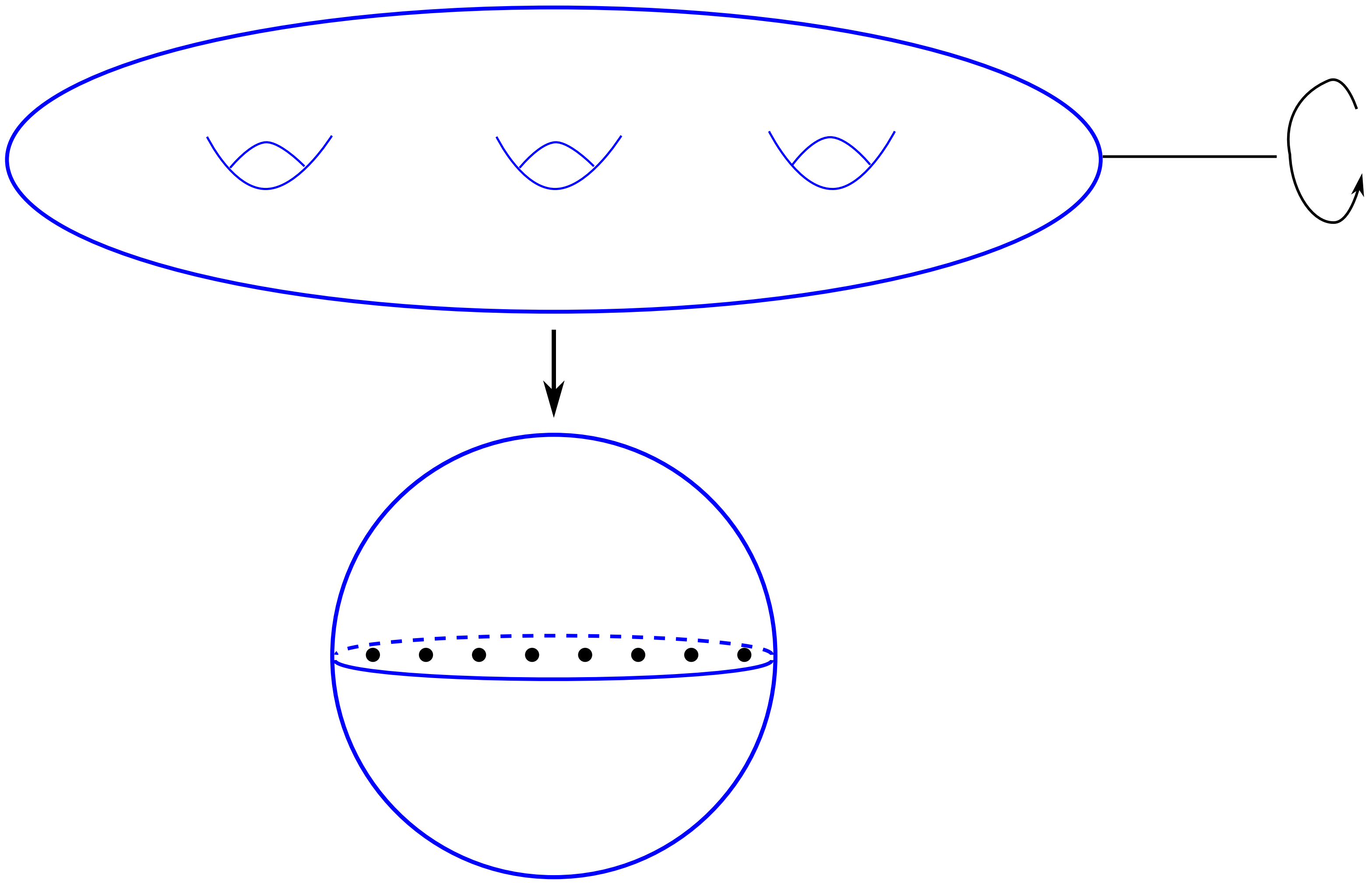
Fact: A homeomorphism lifts if and only if it preserves the set of curves that lift.

Hyperelliptic Involution



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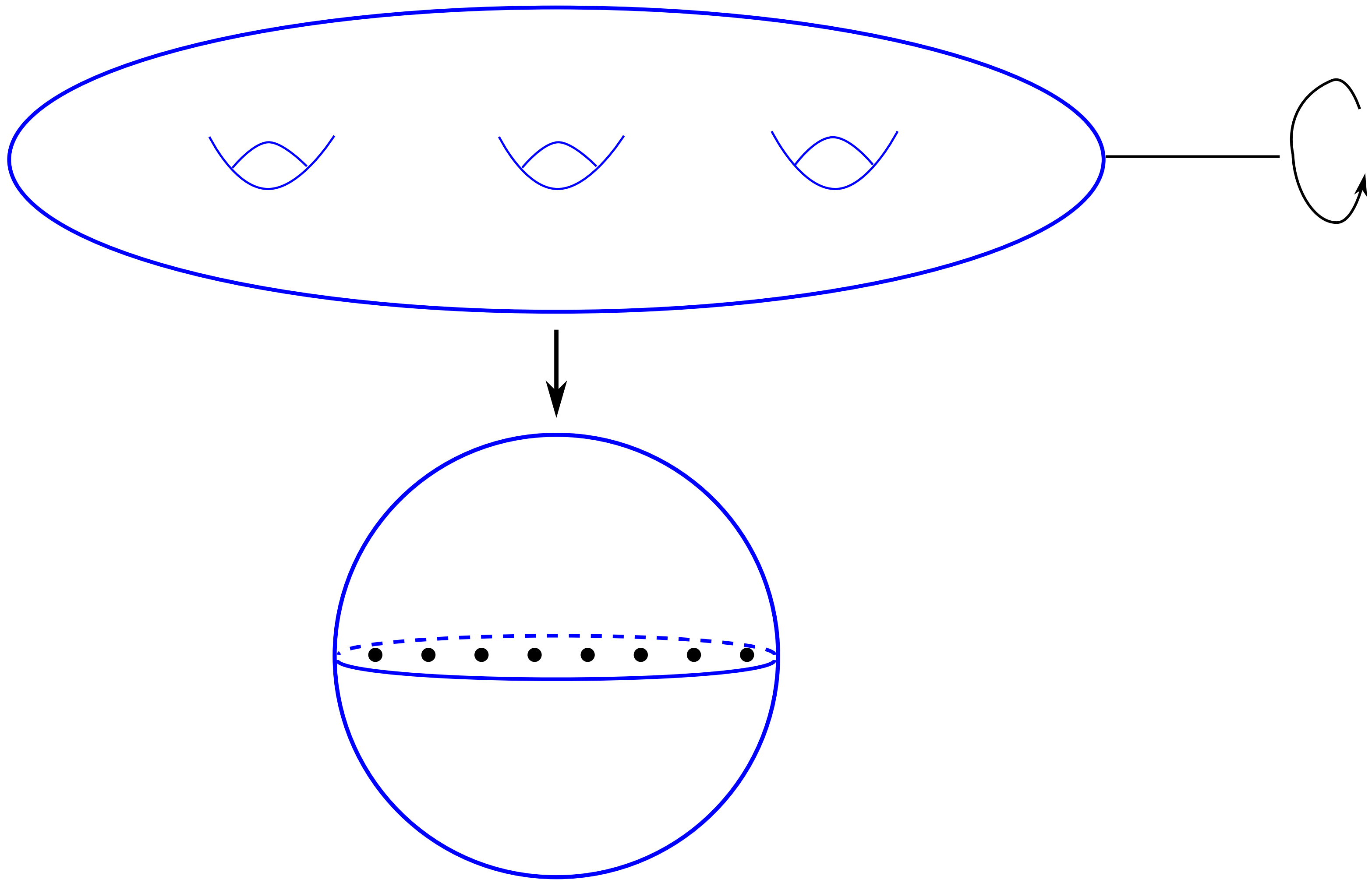
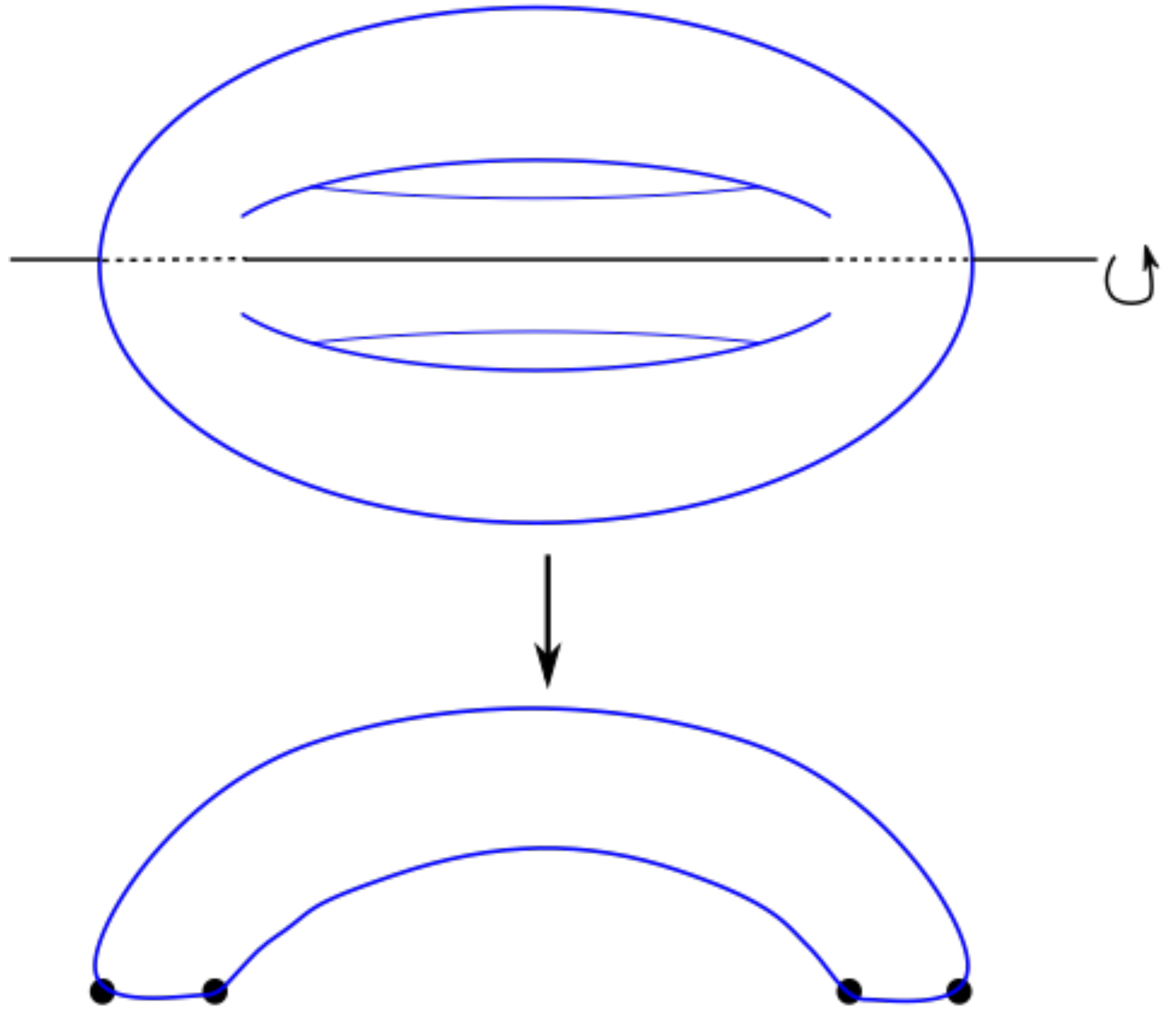


c in S^2 lifts $\Longleftrightarrow$ c bounds an even number of branch points

But wait!



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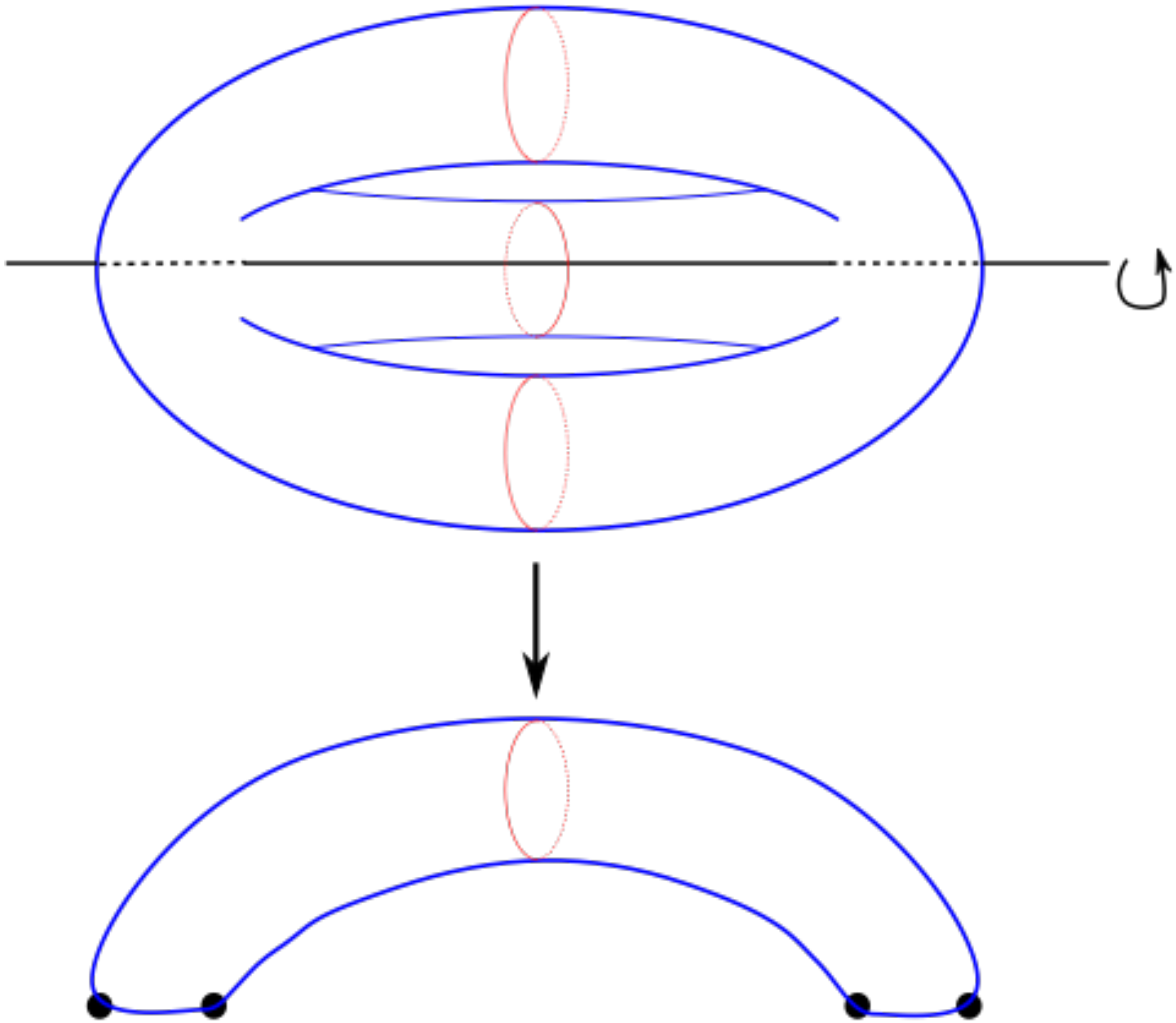

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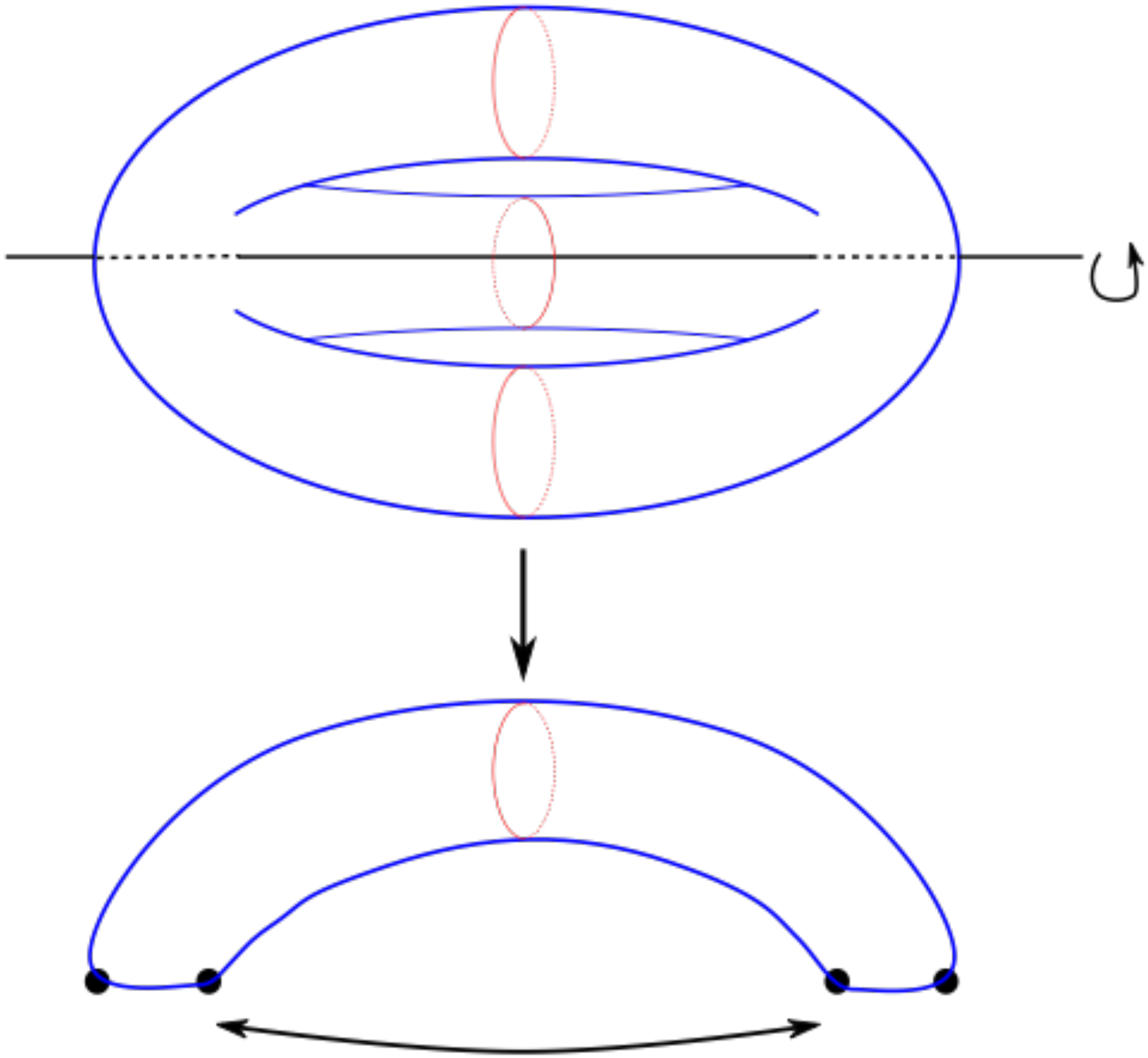

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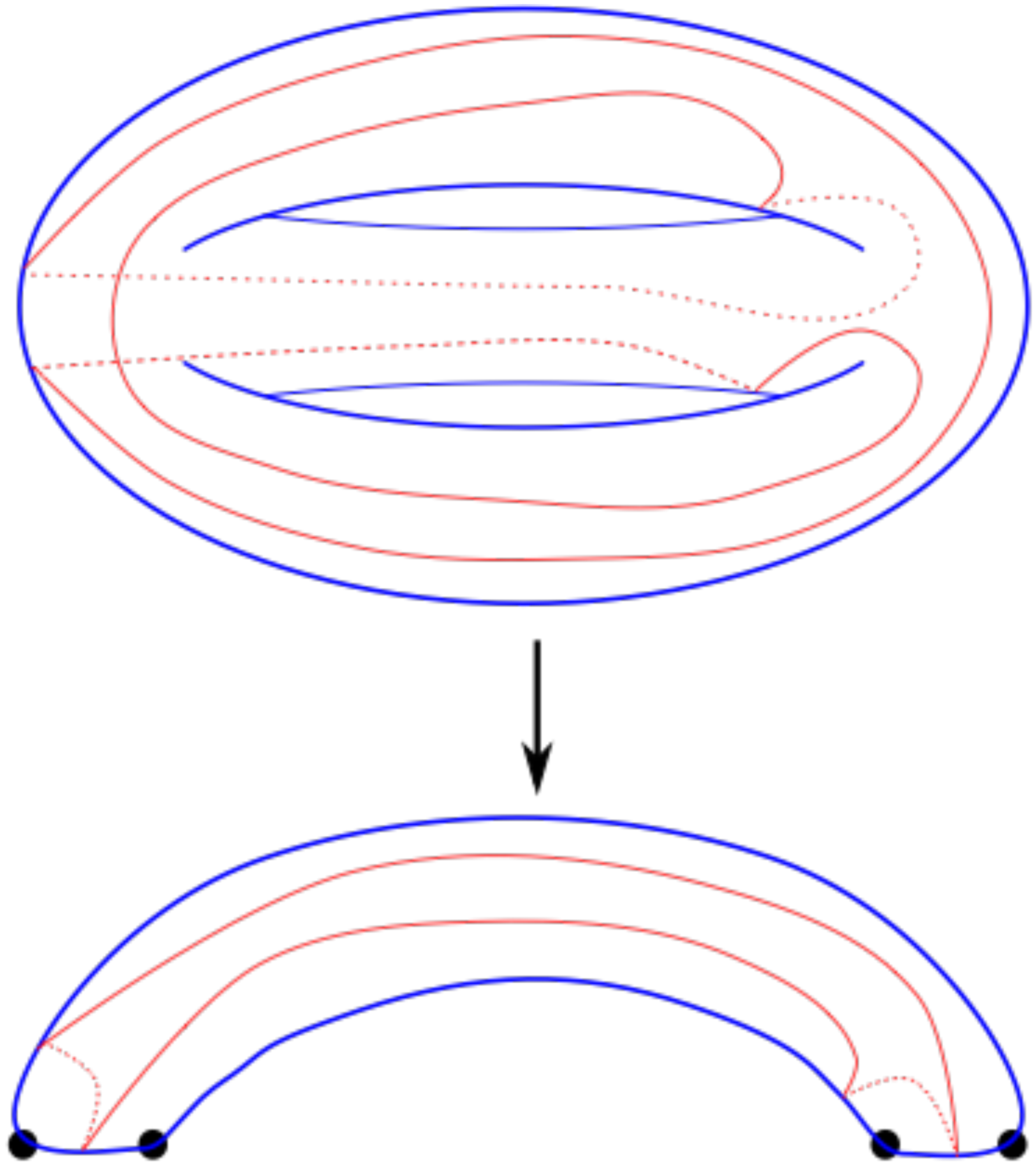

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From Birman—Hilden "On isotopies of homeomorphism of Reimann surfaces." *Annals of Mathematics.*

LEMMA 5.1. *Let* $(p, T_{g,0}, T_{0,0})$ *be a cyclic branched covering. Let* $(\tilde{p}, T_{g,n}, T_{0,n})$ *be the associated unbranched covering. Then every homeomorphism of* $T_{0,n}$ *lifts to a homeomorphism of* $T_{g,n}$. (Remark: the lift is only unique up to covering transformations.)

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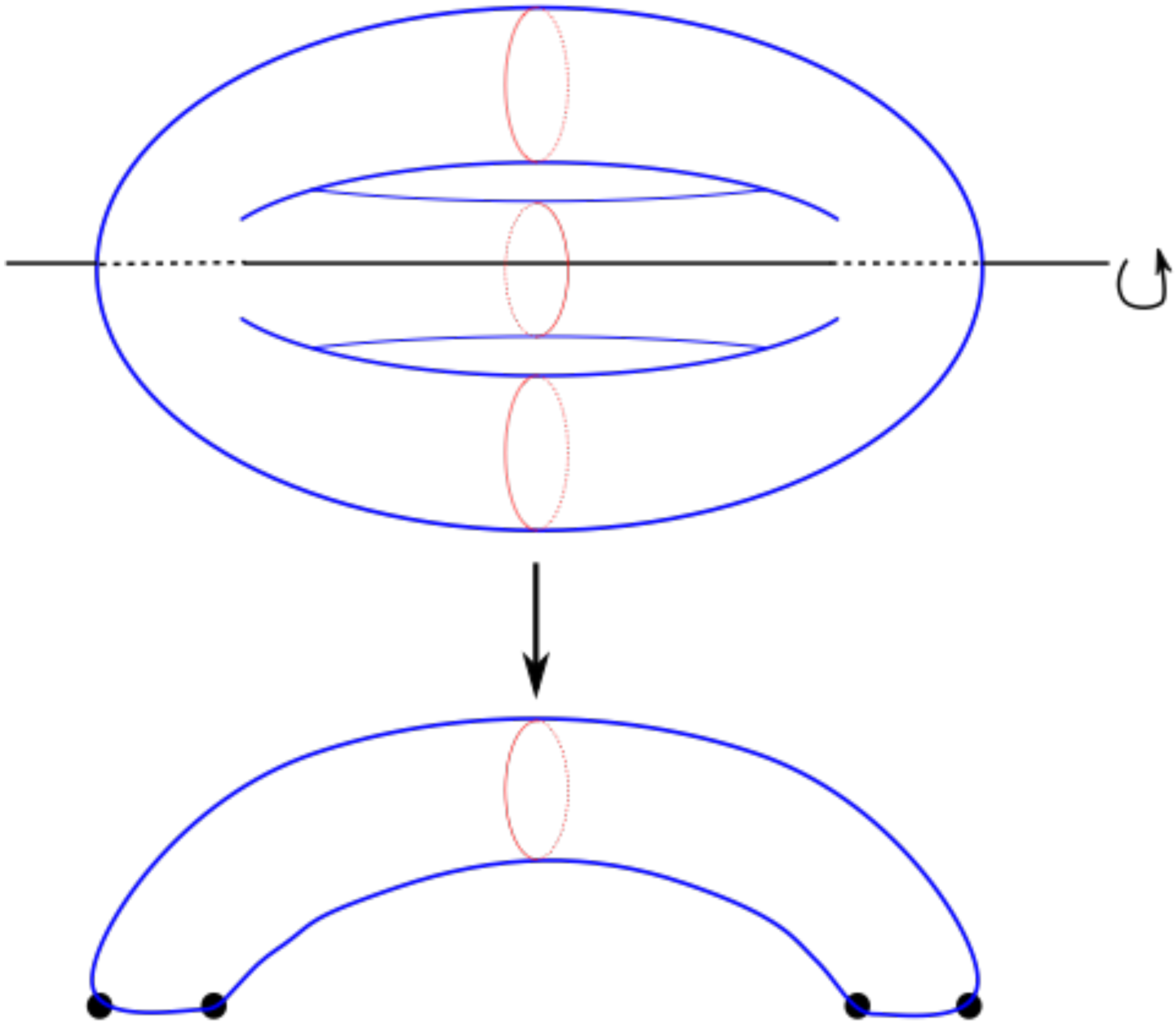
Proof: A homeomorphism lifts if and only it preserves the set of curves that lift.

Since the cover is degree k, a curve lifts if and only if it surrounds k branch points…

A curve that lifts!



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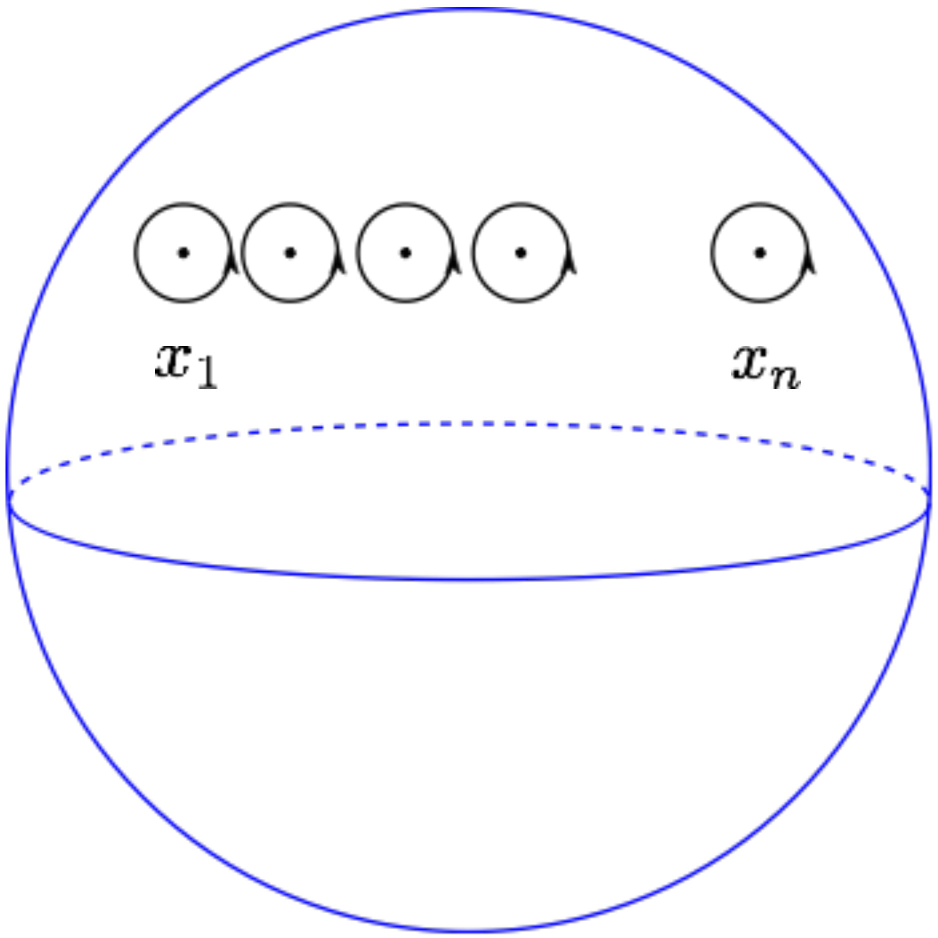
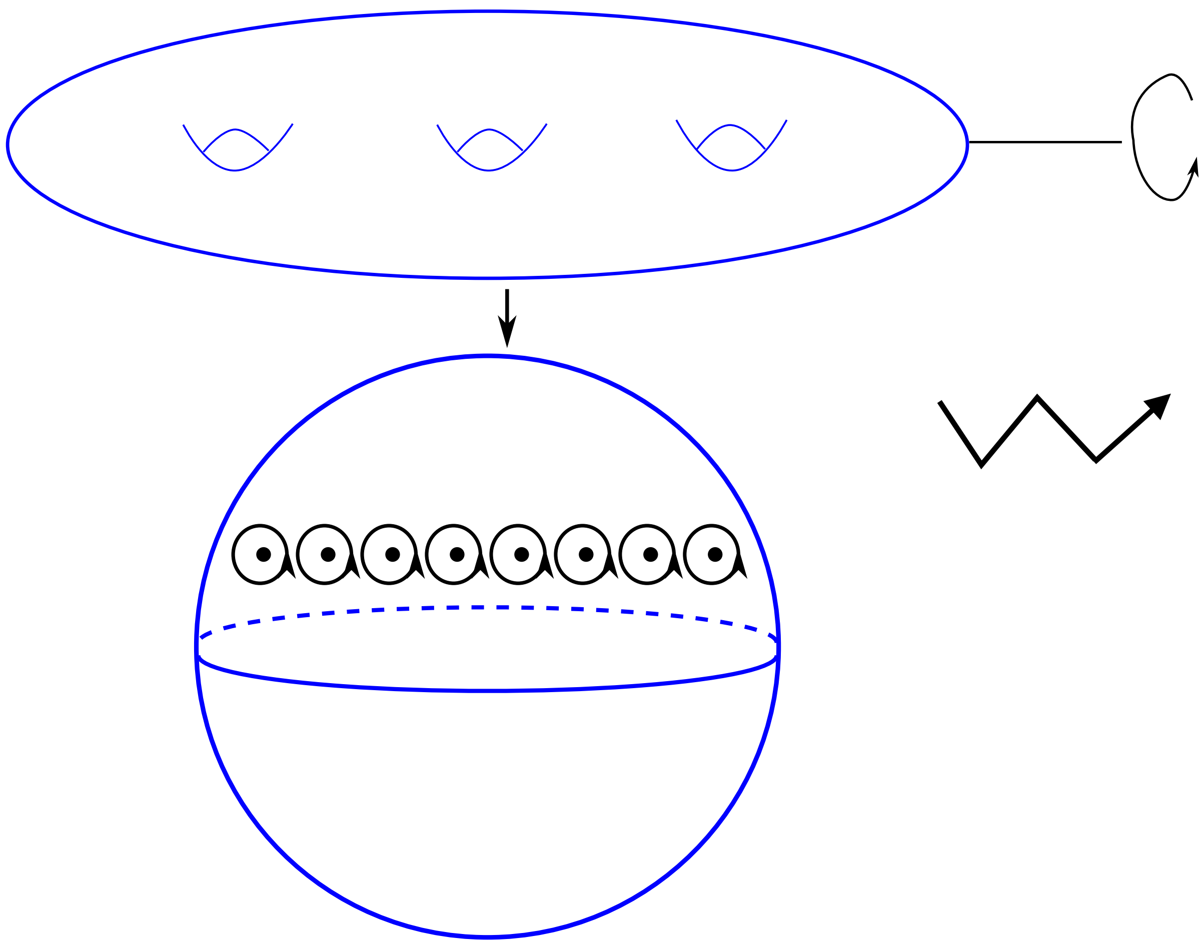
Setup

$p : S_g \to S^2$

$S^{2,n} = S^2 \setminus$ {branch points}



The map on homology



$$\rho : H_1(S_{0,n}, \mathbb{Z}) \to \mathbb{Z}/k\mathbb{Z}$$

Main Theorem

Theorem (Ghaswala—W)

Let $p : S_g \to S^2$ be a cyclic branched cover of degree k.

Every homeomorphism of S^2 lifts exactly when either:

1. There are only 2 branch points*

2. All generators of homology map to same element of $\mathbb{Z}/k\mathbb{Z}$

k=2: Hyperelliptic involution

Mapping class group

The mapping class group of a surface (with branch points) of genus g MCG(g): Isotopy classes of (orientation preserving) homeomorphisms that preserve the set of branch/marked points.

Mapping Class Groups and Covering Spaces

$p : S_g \to S_h$ regular branched covering space

LMCG(p) is a subgroup of MCG(h) containing isotopy classes of homeomorphisms of S_h that lift. *(Liftable MCG)*

SMCG(p)- isotopy classes of p-equivariant homeomorphisms of S_g. Subgroup of MCG(g). *(Symmetric MCG)*

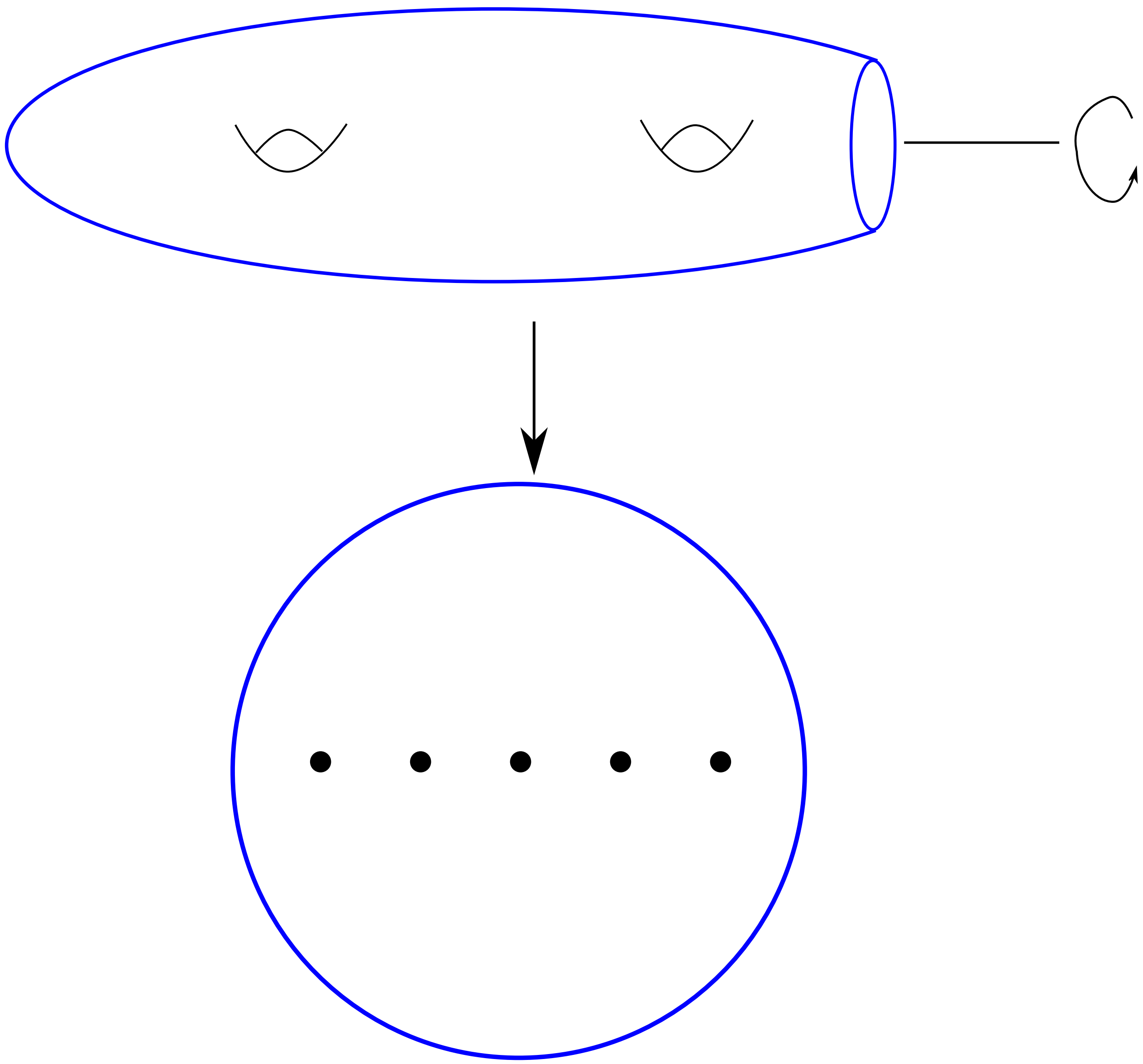
The Birman—Hilden Theorem

Theorem (Birman—Hilden)

$p : S_g \to S_h$ regular branched cover, g>1

SMCG(p)/Deck(p) is isomorphic to LMCG(p).

Important Example



SMCG(p)= B_{2g+1}

Birman—Hilden for Cyclic Covers

Theorem (Birman—Hilden, Ghaswala—Winarski)

$p : S_g \to S^2$ cyclic branched cover, g>1

If all generators of homology map to same element of deck group $\mathbb{Z}/k\mathbb{Z}$, then SMCG(p)/Deck(p) is isomorphic to MCG(0).

Symmetric Mapping Class Group

- Infinite index in MCG
- Finitely presented
- Linear for branched covers of sphere (Bigelow—Budney)
- “Large”? (conjecture of Jain)
    - Image in symplectic group is finite index
- Abelianization of SMCG is isomorphic to the torsion subgroup of an orbifold Picard group if some technical conditions are satisfied (Kordek)