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Lifting Homeomorphisms of Cyclic Branched Covers of the Sphere

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Lifting homeomorphisms of cyclic branched covers of the sphere

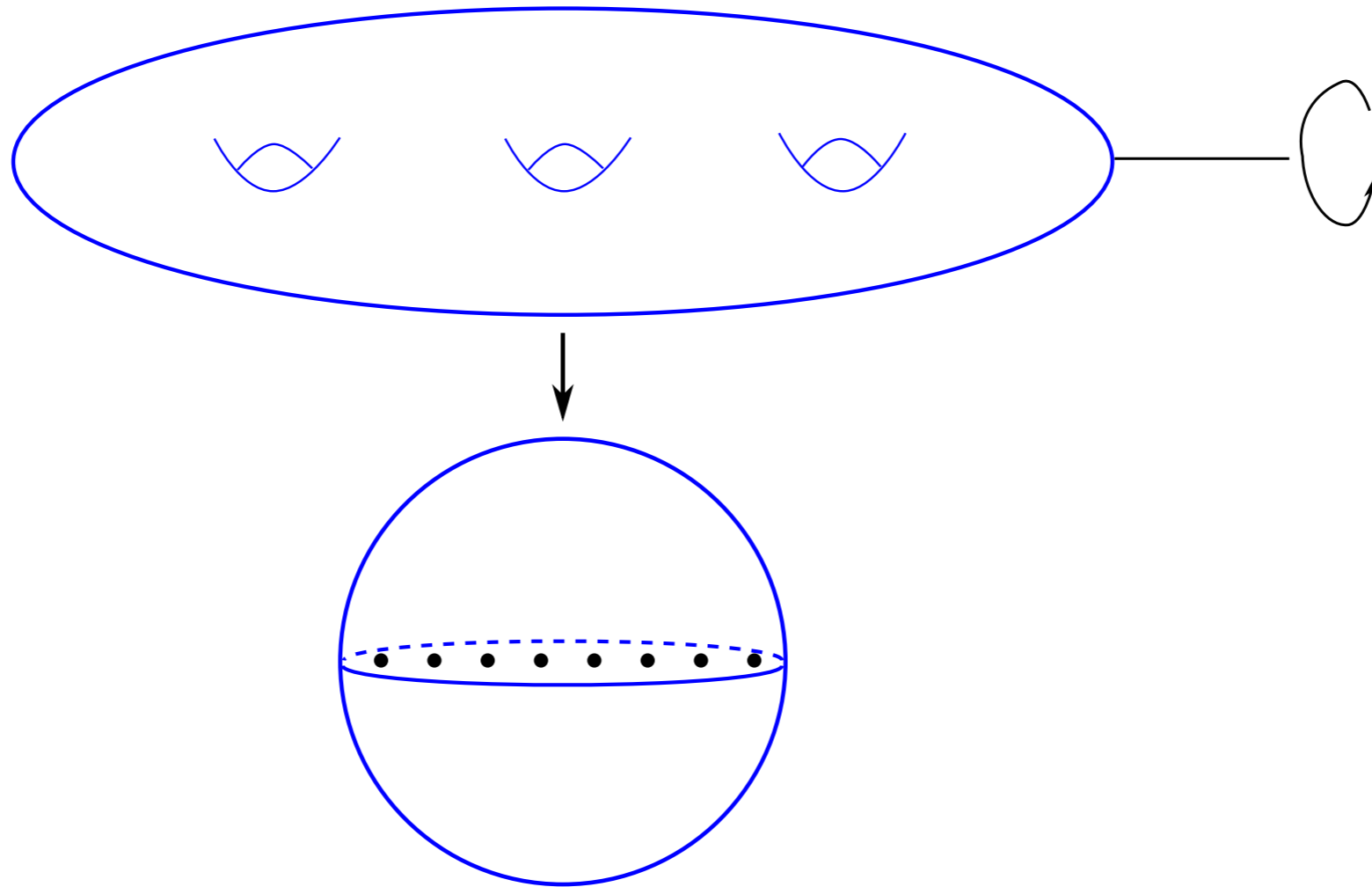
Becca Winarski

University of Wisconsin-Milwaukee

Joint work with Tyrone Ghaswala

Cyclic Branched Covers of the Sphere

$p : S_g \rightarrow S^2$: cyclic branched cover of a sphere.



Cyclic Branched Covers of the Sphere

$$p : S_g \rightarrow S^2$$

Question (Birman—Hilden)

When do all homeomorphisms of S^2 lift?

Cyclic Branched Covers of the Sphere

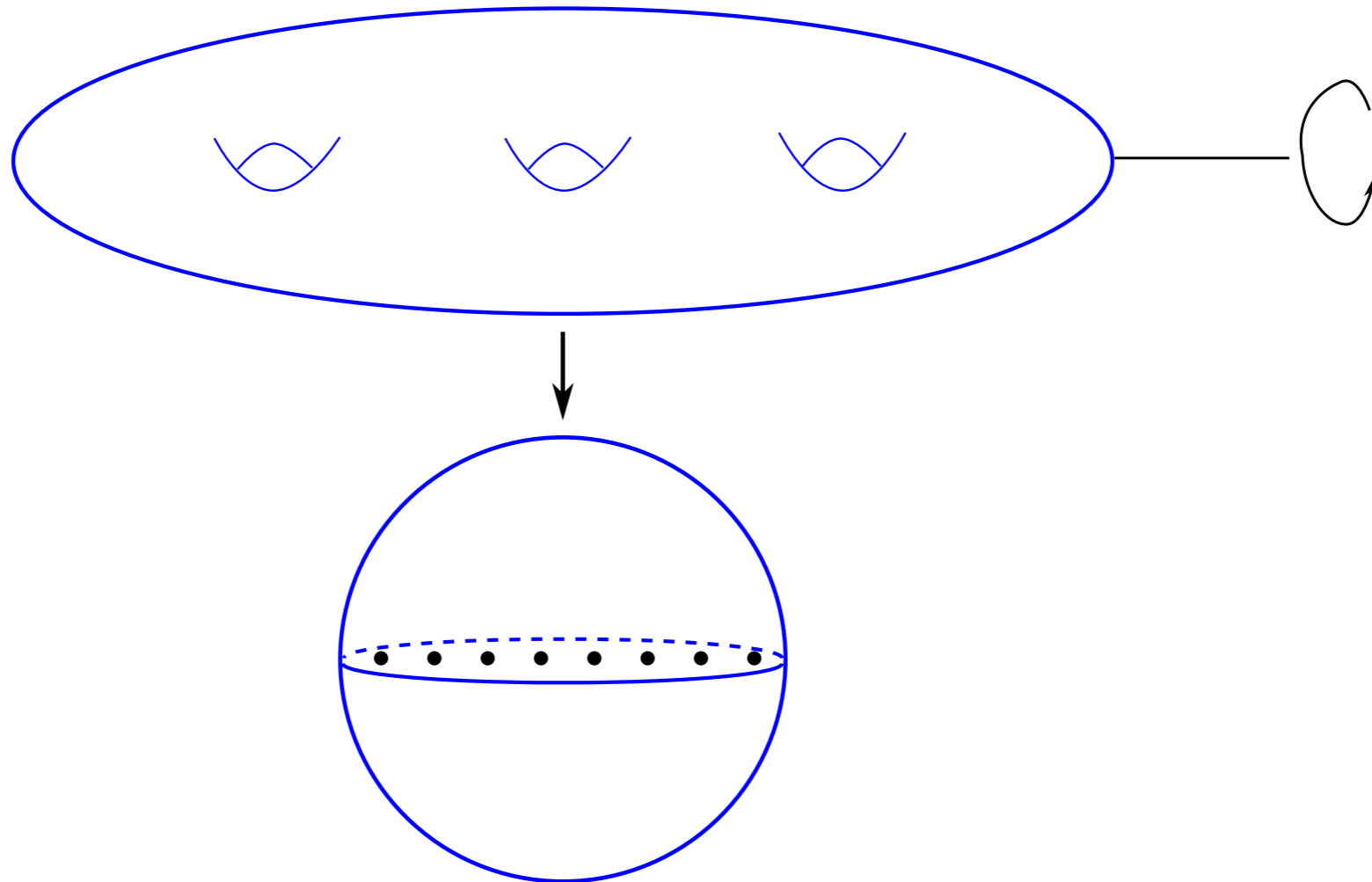
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Question (Birman—Hilden)

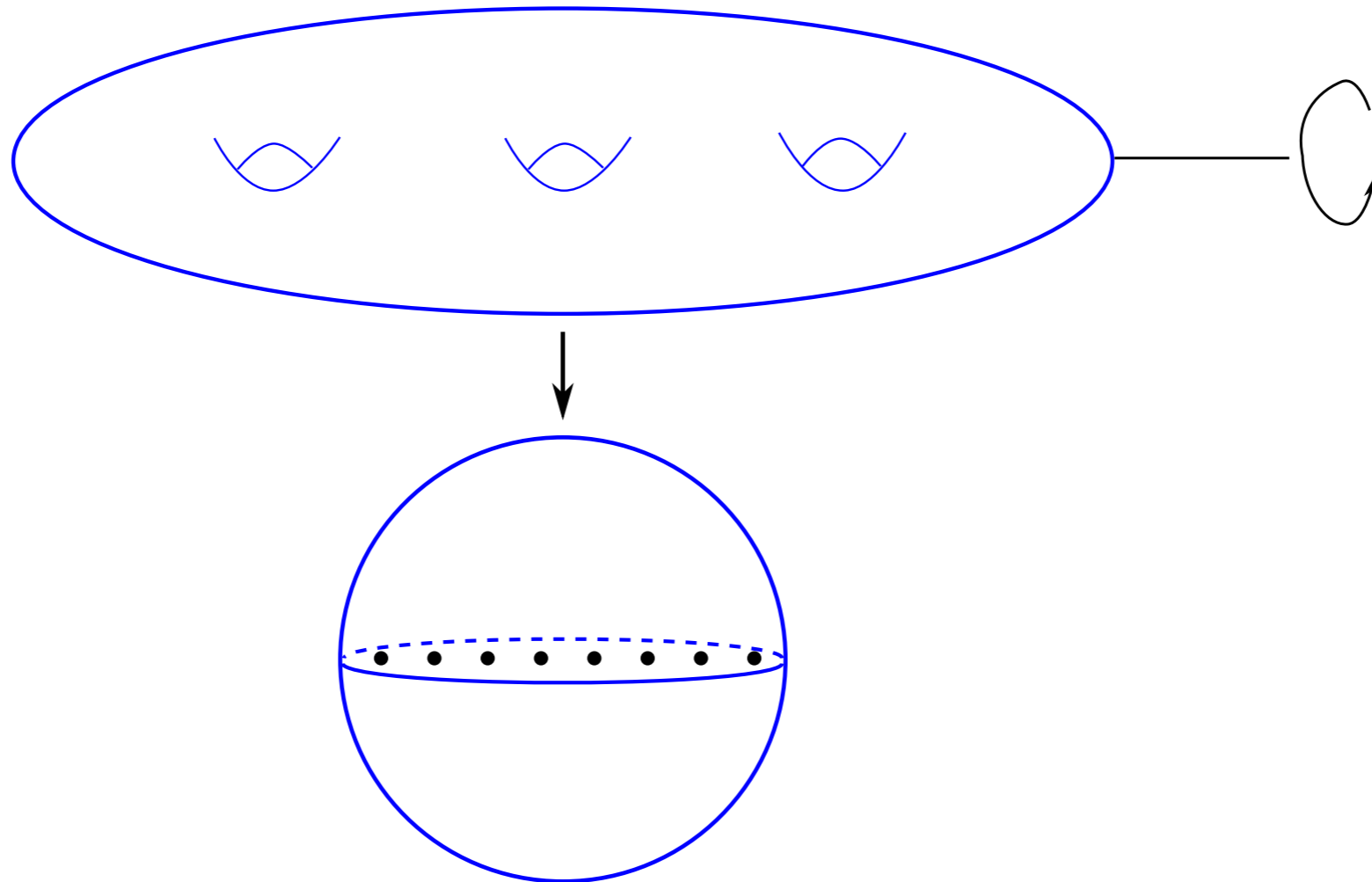
When do all homeomorphisms of S^2 lift?

Fact: A homeomorphism lifts if and only if it preserves the set of curves that lift.

Hyperelliptic Involution

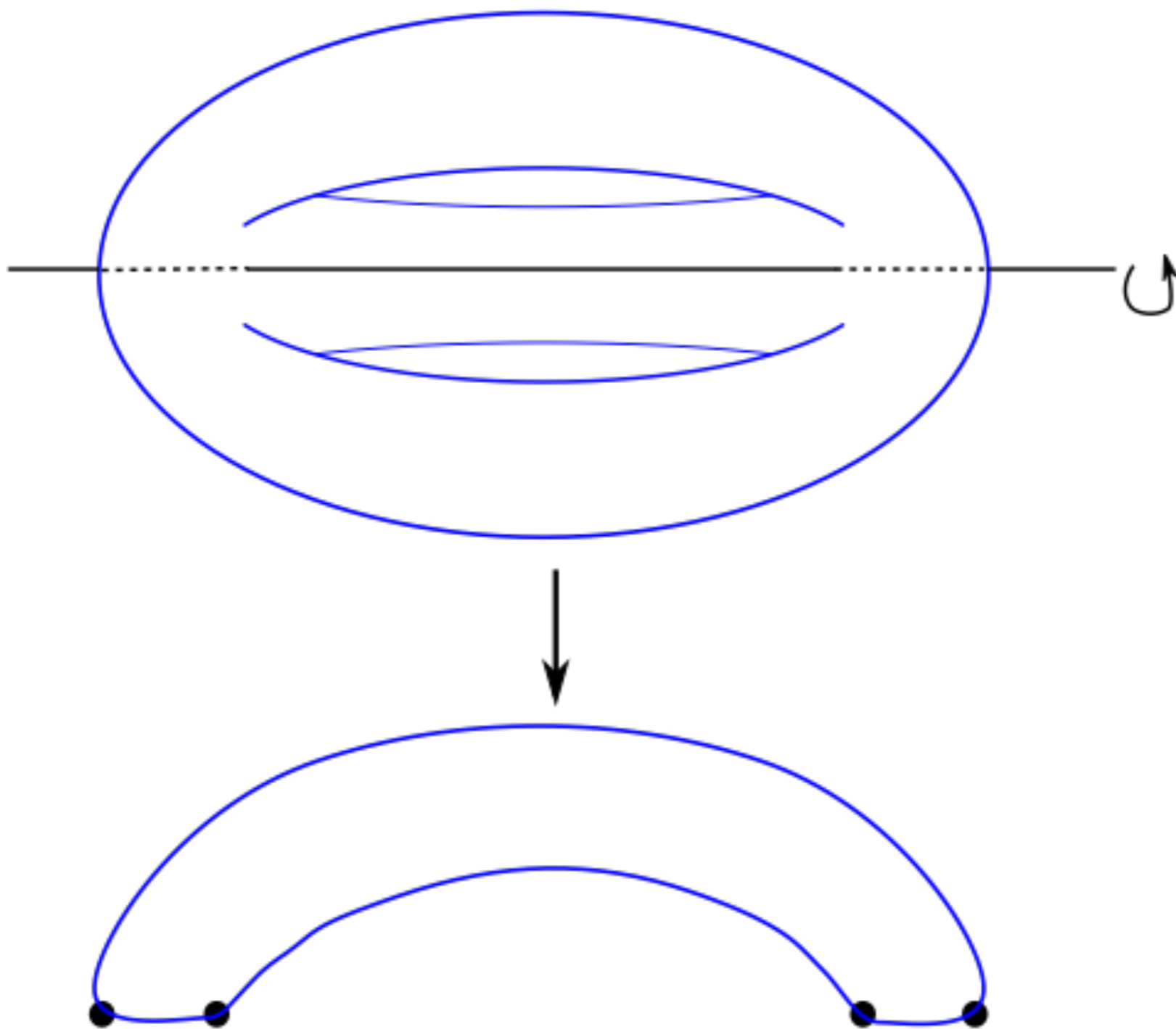


Hyperelliptic Involution

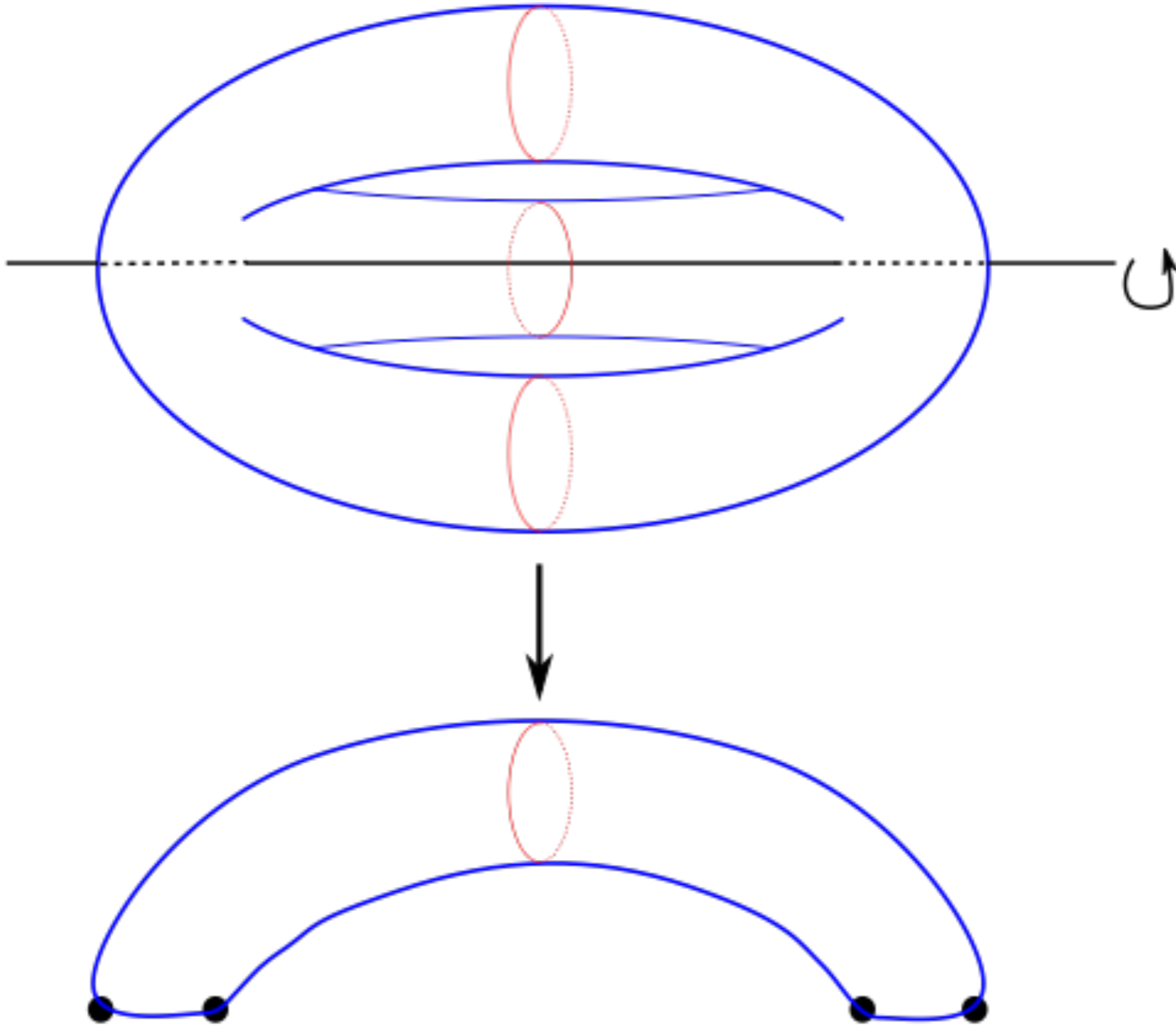


c in S^2 lifts $\iff c$ bounds an even number of branch points

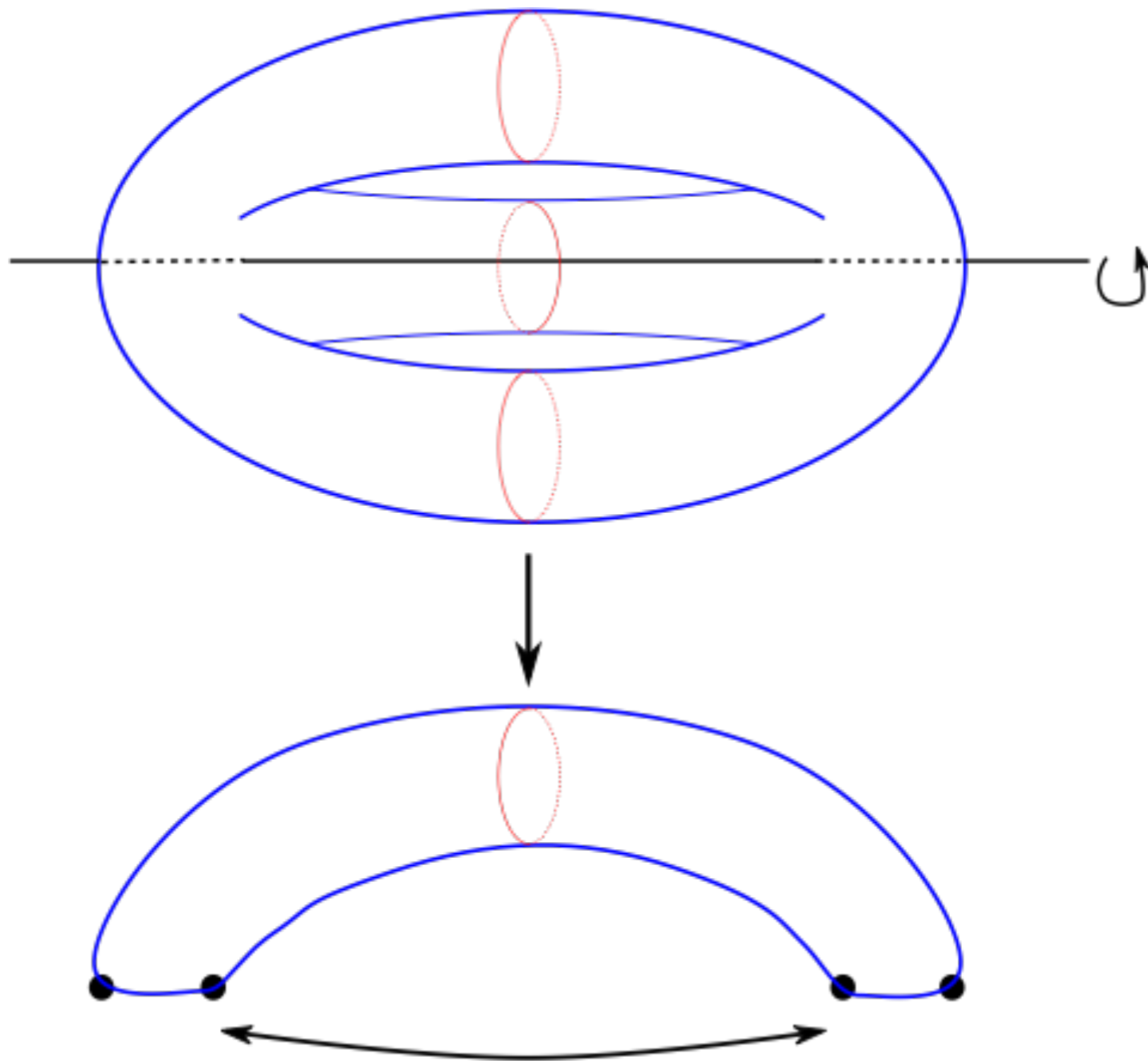
But wait!



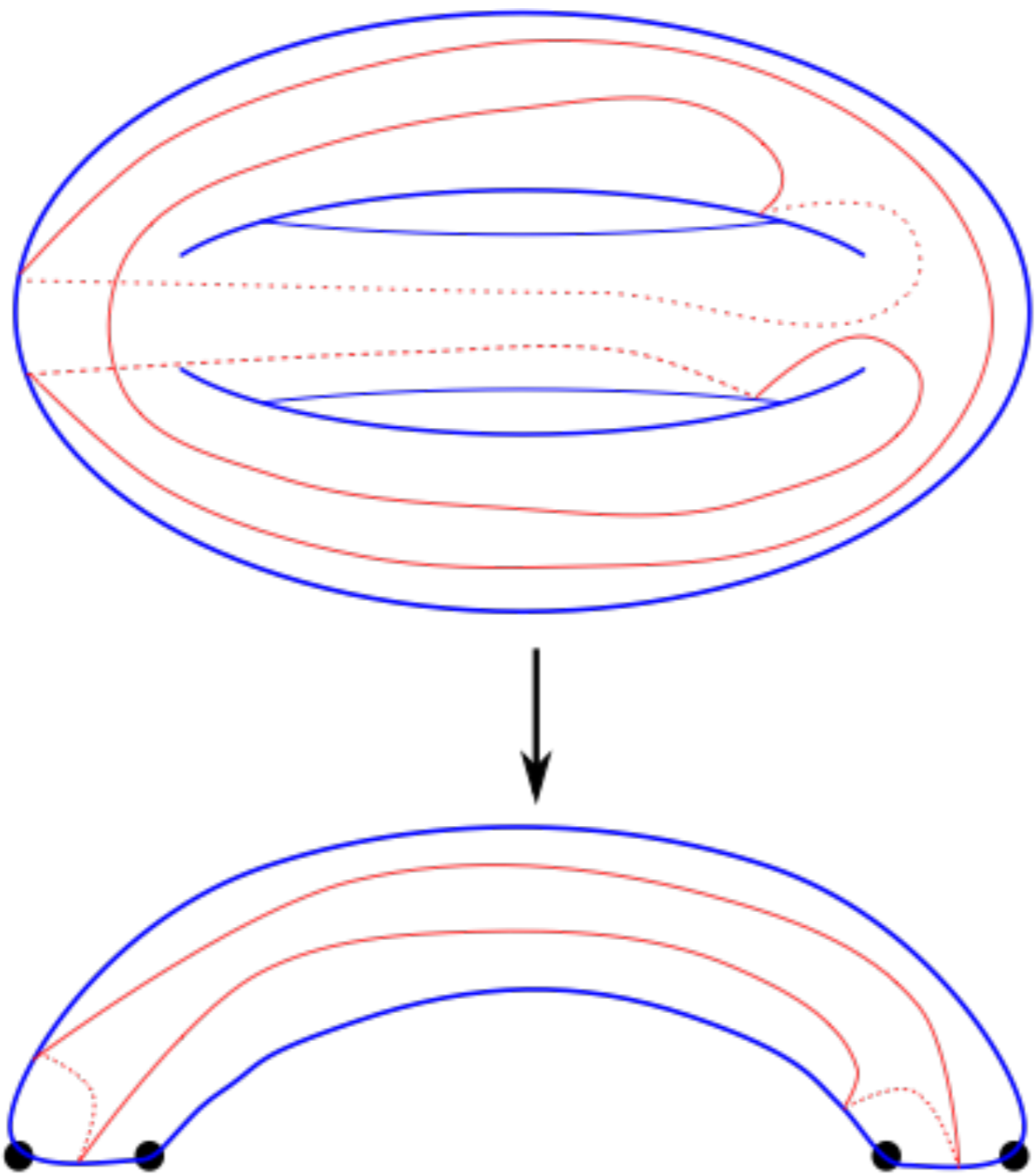
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From Birman—Hilden “On isotopies of homeomorphism of Reimann surfaces.” *Annals of Mathematics*.

LEMMA 5.1. *Let $(p, T_{g,0}, T_{0,0})$ be a cyclic branched covering. Let $(\tilde{p}, T_{g,n}, T_{0,n})$ be the associated unbranched covering. Then every homeomorphism of $T_{0,n}$ lifts to a homeomorphism of $T_{g,n}$. (Remark: the lift is only unique up to covering transformations.)*

But wait!

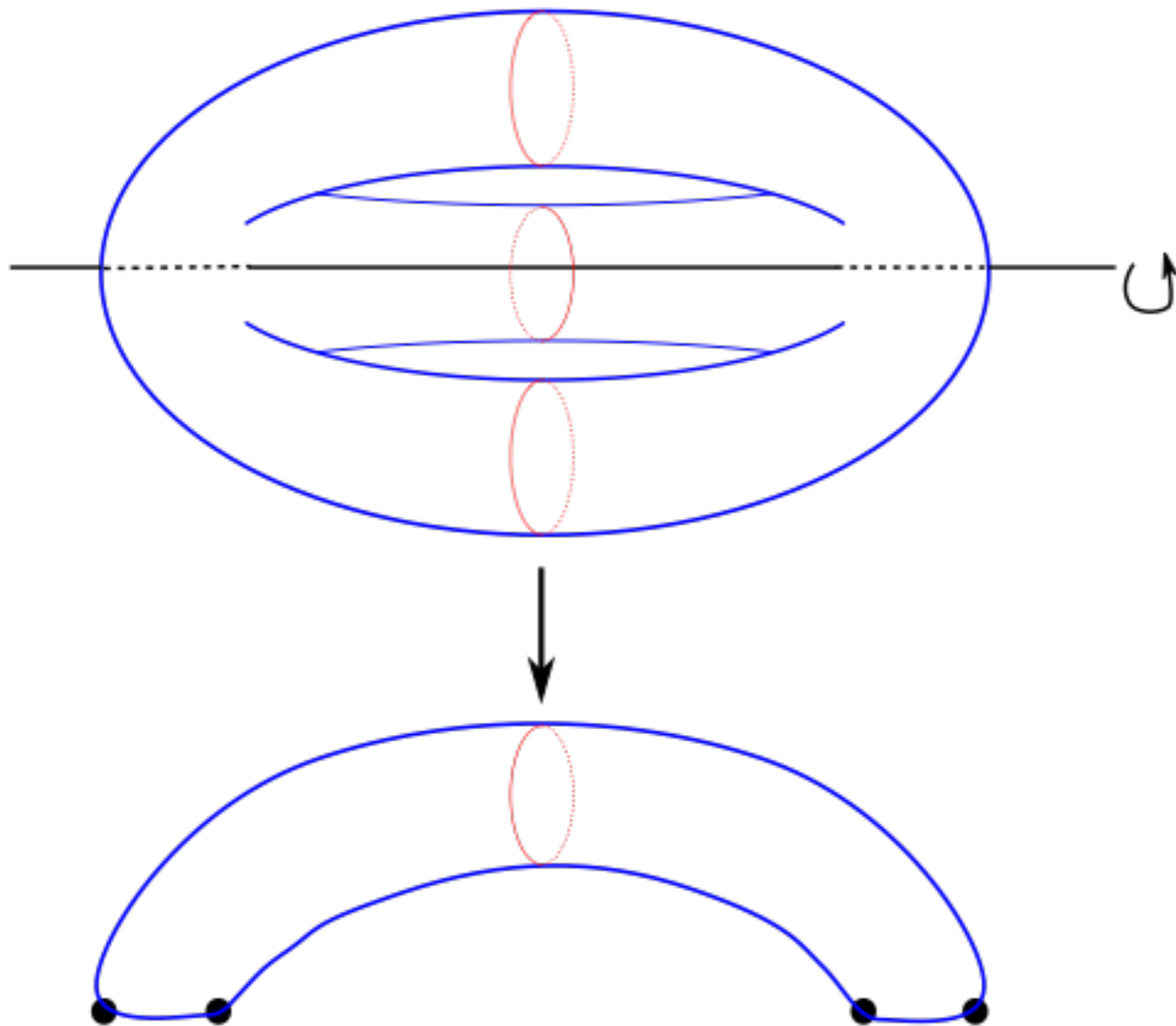
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Proof: A homeomorphism lifts if and only if it preserves the set of curves that lift.

Since the cover is degree k , a curve lifts if and only if it surrounds k branch points...

A curve that lifts!



But wait!

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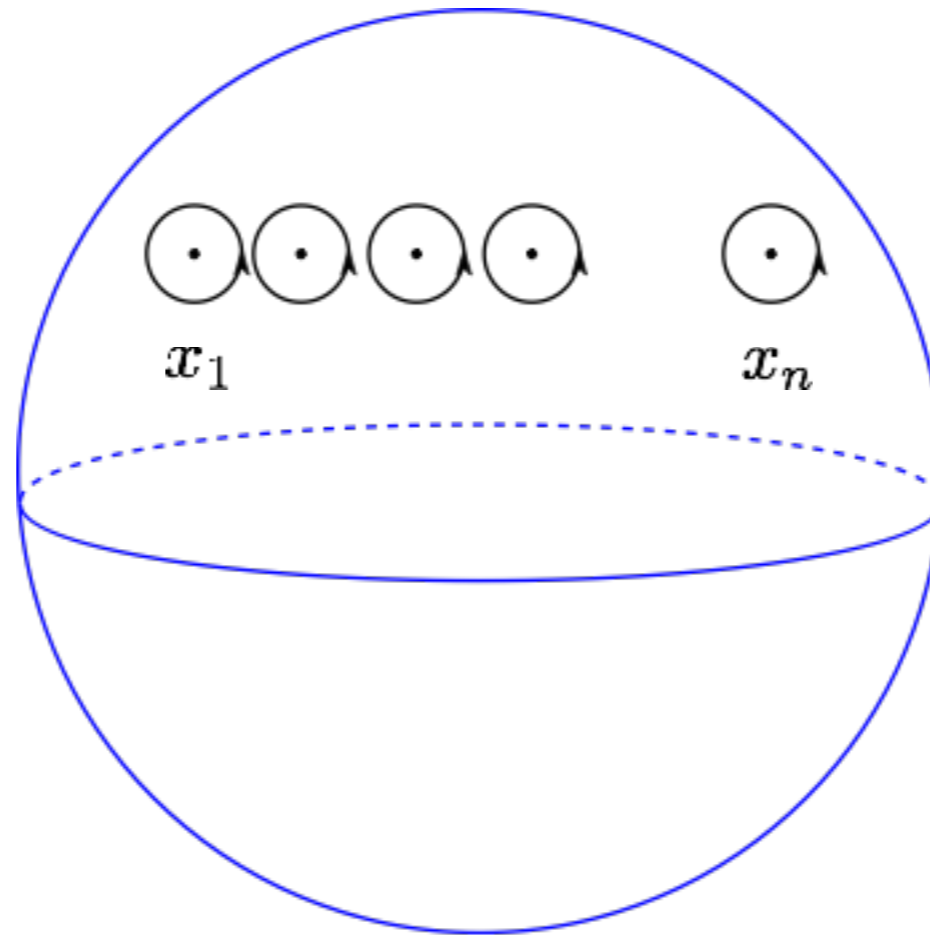
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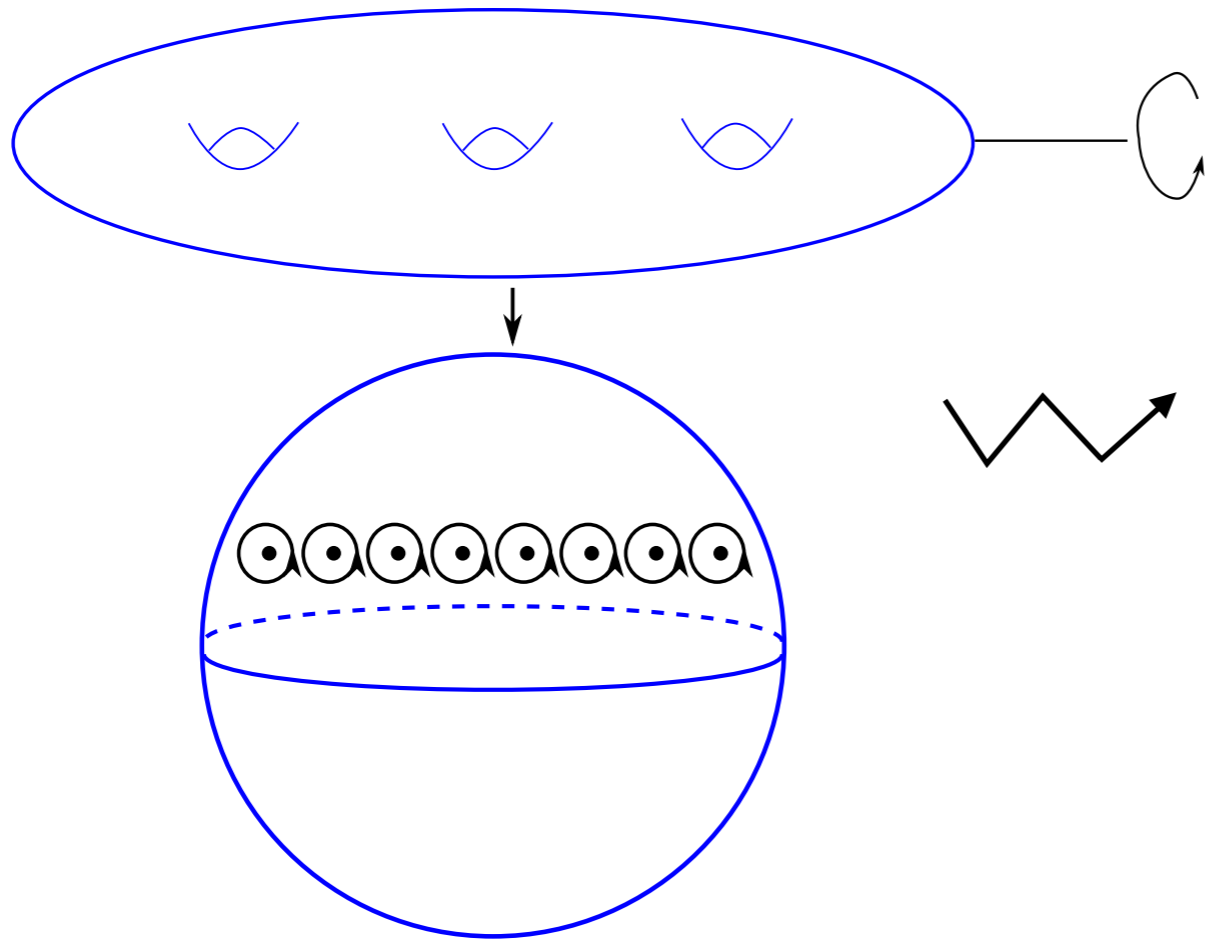
Setup

$$p : S_g \rightarrow S^2$$

$$S^{2,n} = S^2 \setminus \{\text{branch points}\}$$



The map on homology



$\rightsquigarrow \rho : H_1(S_{0,n}, \mathbb{Z}) \rightarrow \mathbb{Z}/k\mathbb{Z}$

Main Theorem

Theorem (Ghaswala—W)

Let $p : S_g \rightarrow S^2$ be a cyclic branched cover of degree k .

Every homeomorphism of S^2 lifts exactly when either:

1. There are only 2 branch points*
2. All generators of homology map to same element of $\mathbb{Z}/k\mathbb{Z}$

$k=2$: Hyperelliptic involution

Mapping class group

The mapping class group of a surface (with branch points) of genus g $MCG(g)$: Isotopy classes of (orientation preserving) homeomorphisms that preserve the set of branch/marked points.

Mapping Class Groups and Covering Spaces

$p : S_g \rightarrow S_h$ regular branched covering space

LMCG(p) is a subgroup of MCG(h) containing isotopy classes of homeomorphisms of S_h that lift. (*Liftable MCG*)

SMCG(p)- isotopy classes of p-equivariant homeomorphisms of S_g . Subgroup of MCG(g). (*Symmetric MCG*)

The Birman—Hilden Theorem

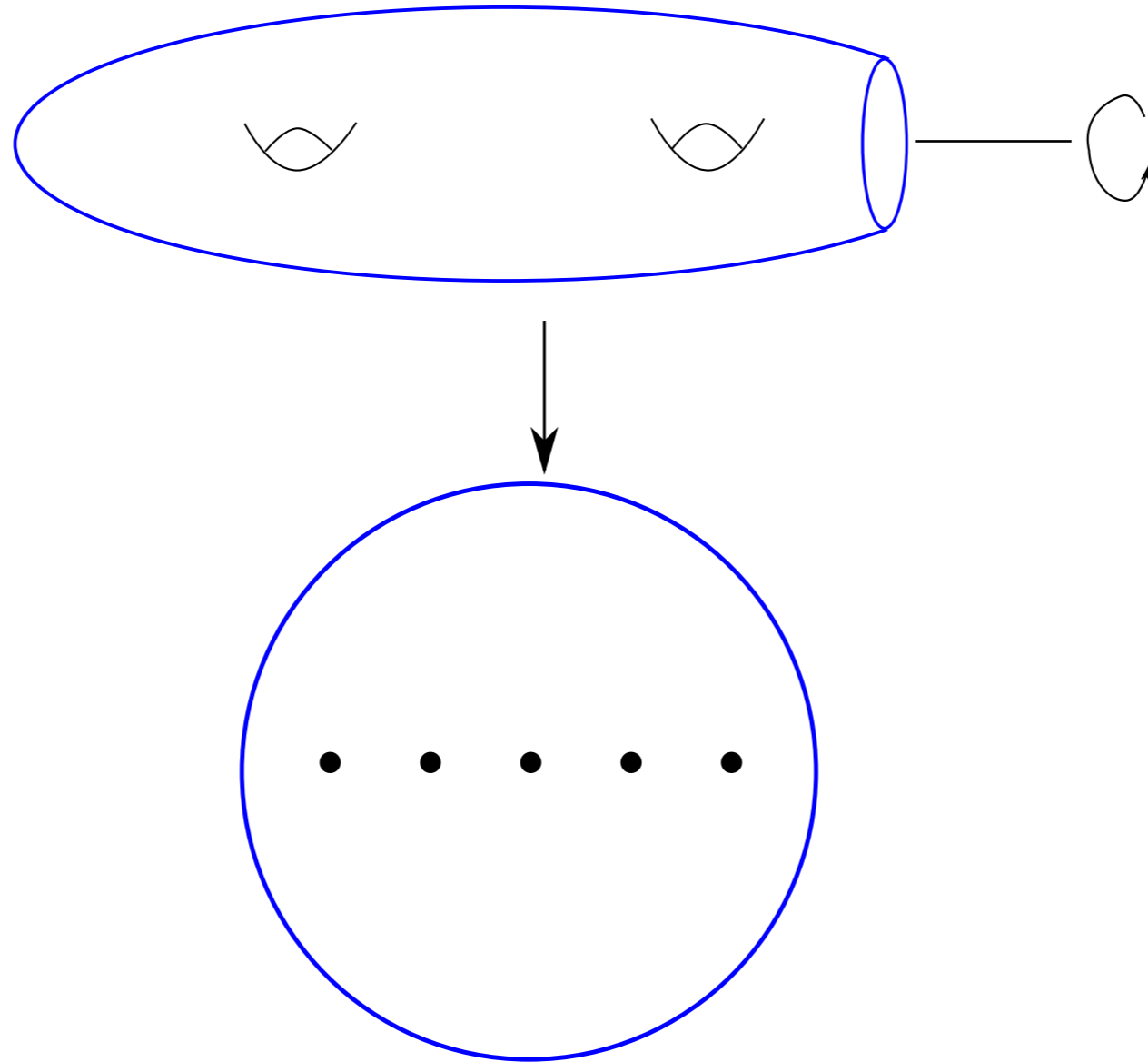
Theorem (Birman—Hilden)

$p : S_g \rightarrow S_h$ regular branched cover, $g > 1$

$\text{SMCG}(p)/\text{Deck}(p)$ is isomorphic to $\text{LMCG}(p)$.



Important Example



$$\text{SMCG}(p) = B_{2g+1}$$

Birman—Hilden for Cyclic Covers

Theorem (Birman—Hilden, Ghaswala—Winarski)

$p : S_g \rightarrow S^2$ cyclic branched cover, $g > 1$

If all generators of homology map to same element of deck group $\mathbb{Z}/k\mathbb{Z}$, then $\text{SMCG}(p)/\text{Deck}(p)$ is isomorphic to $\text{MCG}(0)$.

Symmetric Mapping Class Group

- Infinite index in MCG
- Finitely presented
- Linear for branched covers of sphere (Bigelow—Budney)
- “Large”? (conjecture of Jain)
 - Image in symplectic group is finite index
- Abelianization of SMCG is isomorphic to the torsion subgroup of an orbifold Picard group if some technical conditions are satisfied (Kordek)

