Enriched Topology and Asymmetry

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ENRICHED TOPOLOGY AND ASYMMETRY

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Motivations for Asymmetry

First Motivation: Quasimetric Spaces

Consider possible conditions for \((X, d)\), where \(d : X \times X \to \mathbb{R}\):

- **(M1)** \(\forall x, y \in X, \ d(x, y) \geq 0\) (non-negativity)
- **(M2)** \(\forall x, y \in X, \ [d(x, y) > 0 \text{ or } d(y, x) > 0] \iff x \neq y\) (weak pos. definiteness)
  - **(M2a)** \(\forall x, y \in X, \ d(x, y) = 0\) if \(x = y\) (zero-distance on diagonal)
  - **(M2b)** \(\forall x, y \in X, \ [d(x, y) > 0 \text{ and } d(y, x) > 0] \iff x \neq y\) (strong pos. def.)
- **(M3)** \(\forall x, y \in X, \ d(x, y) = d(y, x)\) (symmetry)
- **(M4)** \(\forall x, y, z \in X, \ d(x, z) \leq d(x, y) + d(y, z)\)

Note: \(M2b \Rightarrow M2, \ M3 \Rightarrow (M2 \iff M2b), \ M1 \Rightarrow (M2 \Rightarrow M2a, \ M2b \Rightarrow M2a), \ M2a \iff s(x, y) = 1 - d(x, y)\) reflexive

**Defn.** \((X, d)\) is: metric space if \(M1, M2, M3, M4\); pseudometric space if \(M1, M2a, M3, M4\) quasimetric space if \(M1, M2b, M4\) hemimetric space if \(M1, M2a, M4\)
**Enriched Topology and Asymmetry**

For hemimetric space \((X, d)\), have topology \(T_d\) generated by \(\varepsilon\)-balls. Then for hemimetric \(d\), \((X, T_d)\) is: \(T_0 \iff M2\); \(T_1 \iff M2b\); \(T_2\) if \(M2\) and \(M3\).

**Second Motivation: Specialization preorders.** Let \((X, T)\) be topological space; put

\[ x \leq_T y \iff y \in \overline{\{x\}} \]

This (or dual) is *specialization* preorder. It is antisymmetric iff \((X, T)\) is \(T_0\); it is antisymmetric and symmetric (and hence equality \(=\)) iff \((X, T)\) is \(T_1\).

Working with sober, non-\(T_1\) spaces means working with asymmetric specialization preorders.

**Defn.** Topological space \((X, T)\) *asymmetric* if specialization order not symmetric.

In practice, asymmetry means \(T_0\) but not \(T_1\) spaces.
**Order-Theoretics: PO-Groupoids, IIA Operators/Involution**

*Groupoid/magma:* $(X, \otimes)$, $\otimes: X \times X \rightarrow X$ ("flat") *semitensor*

*unital:* $(X, \otimes, e)$, two-sided identity $e$

*monoid:* $(X, \otimes, e)$ unital semigroup

*semiquantale:* $(L, \leq, \otimes)$ with $(L, \leq)$ complete lattice, $(L, \otimes)$ groupoid

*Po-groupoid:* $(L, \leq, \otimes)$ with $(L, \leq)$ poset, $(L, \otimes)$ groupoid, $\otimes$ isotone both var.

*IIA (or involutive) po-groupoid:* $(L, \leq, \otimes, *)$, $(L, \leq, \otimes)$ po-groupoid, $*: L \rightarrow L$ involutive, isotone, anti-automorphism in these senses:

  - involutive $(a^{**} = a)$, isotone $(a \leq b \Rightarrow a^* \leq b^*)$,
  - interchanges with $\otimes$ ($(a \otimes b)^* = b^* \otimes a^*$)

*left[right]-residuated po-groupoid:* $(L, \leq, \otimes)$ with $\downarrow [\uparrow]: L \times L \rightarrow L$ s.t.

  $$a \downarrow b \geq c \iff a \otimes c \leq b \quad [b \uparrow a \geq c \iff c \otimes a \leq b]$$

*po-monoid:* $(L, \leq, \otimes)$ unital po-groupoid with $\otimes$ associative
**Enriched Topology and Asymmetry**

*complete po-groupoid*: ordered semiquantale, i.e., \((L, \leq)\) complete lattice

*semiframe/semilocale*: \(\otimes = \wedge\) (binary), integral

*complete po-semigroup*: \(\otimes\) associative

*quantale*: \(\otimes\) distributes on both sides across arbitrary \(\vee\), \(\otimes\) assoc.

*frame/locale*: \(\otimes = \wedge\) (binary), integral

**Semiquantale morphisms**: preserve \(\vee, \otimes\)

**Comment.** Given \((L, \leq, \otimes)\), an IIA-operator/involution may be chosen to be \(\ast = id_L\) iff \(\otimes\) is commutative. Allowable choices of IIA-operators/involutions for \(\otimes\) roughly gauge deviation of \(\otimes\) from being commutative. Non-commutative \(\otimes\) and associated IIA-operators/involutions \(\ast\) tied to possible symmetries of \(L\)-valued specialization orders for \(L\)-topological spaces.
**Example.** \( (L, \leq, ^\prime) \) DeMorgan algebra (complete). Construct \( (S(L), \leq, \circ, *) \) by:

\[
S(L) = \{ f : L \to L \mid f \text{ preserves } \bigvee \},
\leq \text{ taken pointwise,}
\circ \text{ composition of functions,}
f^*(a) = (f^+(a'))',
\]

where \( f^+ : L \to L \) is right adjoint of \( f \) guaranteed by AFT(\( \bigvee \)) and given by

\[
f^+(b) = \bigvee_{f(a) \leq b} a.
\]

Then \( (S(L), \leq, \circ, *) \) is unital, IIA quantale; and it is both non-integral and non-commutative iff \( |L| \geq 3 \).

**Powerset monads and topology.** Fix \( X, Y \) sets, \( L \) semiquantale, \( f : X \to Y \); have \( f_L^- : L^X \to L^Y, f_L^\leftarrow : L^X \leftarrow L^Y, f_L^\rightarrow : L^X \to L^Y \), given respectively by

\[
f_L^-(a)(y) = \bigvee_{f(x) = y} a(x), \quad f_L^\leftarrow(b) = b \circ f, \quad f_L^\rightarrow(a) = \bigwedge_{a \leq f_L^-(b)} b.
\]
Fact $f_L^\uparrow \dashv f_L^\downarrow \dashv f_L\rightarrow$

$(X, \tau)$ is L-topological space if $\tau \subset L^X$ closed under $\bigvee, \otimes, \underline{\top}$. $f : (X, \tau) \rightarrow (Y, \sigma)$ L-continuous if $\forall v \in \sigma, f_L^\downarrow(v) \in \tau$.

Fact L-Top topological construct.
Sets Enriched by PO-Monoids

Enriched category $\mathcal{C}$ over a monoidal category $(\mathcal{M}, \otimes, I, a, \lambda, \rho)$ is class of objects with data $C0$, $C1$, and $C2$ subject to axioms D1, D2, and D3:

C0: $\forall a, b \in \mathcal{C}, \exists! \mathcal{C}(a, b) \in |\mathcal{M}|$ (existence of hom-objects)
C1: $\forall a \in \mathcal{C}, \exists id_a : I \rightarrow \mathcal{C}(a, a)$ (existence of identities)
C2: $\forall a, b, c \in \mathcal{C}, \exists ! \circ_{abc} : \mathcal{C}(b, c) \otimes \mathcal{C}(a, b) \rightarrow \mathcal{C}(a, c)$ (comp. of hom-objects)

D1: $\forall a, b, c, d \in \mathcal{C}, \big((\circ_{abd}) \circ (\circ_{bcd} \otimes 1_{\mathcal{C}(a,b)})\big) = \big((\circ_{acd}) \circ (1_{\mathcal{C}(c,d)} \otimes \circ_{abc})\big) \circ \alpha$

D2: $\forall a, b \in \mathcal{C}, \lambda = (\circ_{abb}) \circ (id_b \otimes 1_{\mathcal{C}(a,b)})$

D3: $\forall a, b \in \mathcal{C}, \rho = (\circ_{aab}) \circ (1_{\mathcal{C}(a,b)} \otimes id_a)$

Comment. Po-monoid $(L, \leq \otimes)$, taken as a preordered category, is a monoidal category in which $\otimes$ is the categorical tensor product, $I$ is the unit $e$, and the associator $\alpha$ and the unitors $\lambda, \rho$ are all identities.
Prop. If \((L, \leq \otimes)\) is po-monoid, then set \(X\) replacing \(C\) is \(L\)-enriched category iff there is enrichment relation, or \(L\)-(valued) preorder \(P\) on \(X\) s.t.:

E0: \(P : X \times X \to L\) is a mapping (degrees of comparison/precedence)
E1: \(\forall x \in X, e \leq P(x, x)\) (reflexivity)
E2: \(\forall x, y, z \in X, P(y, z) \otimes P(x, y) \leq P(x, z)\) (transitivity)

Defn. For \((L, \leq \otimes)\) a unital po-groupoid, \((X, P)\) is an \(L\)-preordered set, or \((X, L, P)\) is a preordered set, if:

P0: \(P : X \times X \to L\) is a mapping (degrees of comparison/precedence)
P1: \(\forall x \in X, e \leq P(x, x)\) (reflexivity)
P2: \(\forall x, y, z \in X, P(x, y) \otimes P(y, z) \leq P(x, z)\) (transitivity)

Defn. Given \(\mathcal{M}\)-enriched categories \(C, D, F : C \to D\) is \(\mathcal{M}\)-enriched functor if:

F1: \(\forall a \in C, \exists! F(a) \in D\)
F2: \(\forall a, b \in C, \exists! F_{ab} \in \mathcal{M}(C(a, b), D(F(a), F(b)))\)
F3: \(\forall a \in C, F_{aa} \circ id_a = id_{F(a)}\) (in \(\mathcal{M}\))
F4: \(\forall a, b, c \in C, F_{ac} \circ (\circ_{abc}) = (\circ_{F(a)F(b)F(c)}) \circ (F_{bc} \otimes F_{ab})\) (in \(\mathcal{M}\))
**Enriched Topology and Asymmetry**

**Prop.** Let $L$ be po-monoid. Given $L$-preordered sets $(X, P)$, $(Y, Q)$ taken as $L$-enriched categories, then $f : (X, P) \to (Y, Q)$ is $L$-enriched functor iff $f : X \to Y$ is mapping such that

$$P(x, y) \leq Q(f(x), f(y)).$$

**Defn.** For $(L, \leq \otimes)$ a unital po-groupoid, an $L$-isotone map $f : (X, P) \to (Y, Q)$ is a mapping $f : X \to Y$ such that $P(x, y) \leq Q(f(x), f(y))$.

**Defn (fixed-basis).** For unital po-groupoid $L$, the category $L$-$PreSet$ comprises $L$-preordered sets and $L$-isotone mappings together with the composition and identities of $Set$.

**Theorem (fixed-basis).** For each complete unital po-groupoid, the category $L$-$PreSet$ is a topological category over $Set$ w.r.t. expected forgetful functor; i.e., each $L$-$PreSet$ is a topological construct. Hence so is $PreSet$.

**Comment.** Variable-basis enriched functors can be defined for enriched category theory, with corresponding variable-basis morphisms between preordered sets $(X, L, P)$. Topologicity quite delicate; see DMR FSS 2014.
Many-Valued Enriched Topological Systems & Their Extent Spaces

**Question.** Suppose bitstring $x$ precedes bitstring $y$ to some degree $\alpha$, and bitstring $y$ satisfies predicate $a$ to some degree $\beta$. How should bitstring $x$ satisfy predicate $a$ to at least some degree related to both $\alpha$ and $\beta$? What if conjunction of predicates is non-commutative? Potential applications in data-mining and pattern-matching.

**Complete po-groupoid based topological systems.** Let $A, L$ be complete po-groupoids. An $L$-*topological system* $(X, A, \equiv)$ comprises: set $X$, and many-valued *satisfaction relation* $\equiv : X \times A \to L$ which respectively satisfies *join, pretensor, top interchange laws*:

$$\forall x \in X, \forall \{a_\gamma\}_{\gamma \in \Gamma} \subseteq A, \quad \equiv \left( x, \bigvee_{\gamma \in \Gamma} a_\gamma \right) = \bigvee_{\gamma \in \Gamma} \equiv (x, a_\gamma),$$

$$\forall x \in X, \forall a, b \in A, \quad \equiv (x, a \otimes b) = \equiv (x, a) \otimes \equiv (x, b),$$

$$\forall x \in X, \quad \equiv (x, \top) = \top.$$

**Note.** $\forall x \in X, \quad \equiv (x, \bot) = \bot$; i.e., each bitstring never satisfies *false*. 

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**ENRICHED TOPOLOGY AND ASYMMETRY**

**Slide 12**
**Extents of predicates.** Let \((X, A, \models)\) be \(L\)-topological system. The \(L\)-extent operator \(ext_L : A \rightarrow L^X\) given by

\[
\text{ext}_L(a)(x) = \models (x, a)
\]

Then \(ext_L\) preserves arbitrary \(\bigvee, \bigotimes, \top\) (to \(\bigvee\)) from \(A\) to \(L^X\). So extent space

\[
(X, (\text{ext}_L)^\top (A))
\]

is \(L\)-topological space as defined above.

**Notes.**

1. Each object of \(L\text{-Top}\) is produced in this way, i.e., is extent space.
2. Information (Chu) systems named by the character of their extent spaces; above justifies name "\(L\)-topological systems".
PO-Ringoids. $(L, \leq, \otimes, \bigotimes)$ is *po-ringoid* means:

R1: $(L, \leq, \otimes)$ po-groupoid with pretensor $\otimes$
R2: $(L, \leq, \otimes)$ po-groupoid with multiplication $\otimes$
R3: *left-partial distributive law* of $\otimes$ over $\bigotimes$ holds:

$$a \otimes (b \otimes c) \leq (a \otimes b) \otimes (a \otimes c)$$

**Idea.** Want to add topological system structure or topological structure to preset; $(L, \leq, \otimes)$ with completeness and unit undergirds order, $(L, \leq, \otimes)$ with completeness undergirds topology; left-partial distributive law ties together.

**Note.** May add conditions to po-ringoids. *Groupoidal quantale*: unital complete po-ringoid $(L, \leq, \otimes, \bigotimes, e)$ with $(L, \leq, \otimes, e)$ unital quantale and $\otimes$ distributes partially over $\bigotimes$ from both left and right.

**Some Examples.**
(1) Complete lattice $(L, \leq, \lor, \land, \bot)$ unital complete po-ringoid; if also distributive, then $(L, \leq, \land, \lor, \top)$ integral complete po-ringoid.
(2) $([0, 1], \leq, t_L, \land, 1)$ integral groupoidal quantale.
(3) $(S(L), \leq, \circ, \otimes_1, id_L), (S(L), \leq, \circ, \otimes_2, id_L)$ non-integ. groupoidal quantales^note.
**Enriched Topology and Asymmetry**

**Enriched topological systems.** Let \((L, \leq, \otimes, \otimes, e)\) be a unital complete po-ringoid and \((A, \leq, \otimes)\) be a complete po-groupoid. An *L-enriched topological system* \((X, P, A, \models)\) comprises the following:

**ES1:** \(L\)-preordered set \((X, P)\) using unital po-groupoid \((L, \leq, \otimes)\)

**ES2:** \(L\)-topological system \((X, A, \models)\), where satisfaction relation \(\models : X \times A \to L\) respectively satisfies join, *pretensor-multiplication*, top interchange laws using complete po-groupoids \((L, \leq, \otimes)\) and \((A, \leq, \otimes)\)—

\[
\forall x \in X, \forall \{a_\gamma\}_{\gamma \in \Gamma} \subseteq A, \quad \models (x, \bigvee_{\gamma \in \Gamma} a_\gamma) = \bigvee_{\gamma \in \Gamma} \models (x, a_\gamma),
\]

\[
\forall x \in X, \forall a, b \in A, \quad \models (x, a \otimes b) = \models (x, a) \otimes \models (x, b),
\]

\[
\forall x \in X, \quad \models (x, T) = T.
\]

**ES3:** \(P\) and \(\models\) together satisfy *compatibility/enrichment axiom*:

\[
\forall x, y \in X, \forall a \in A, \quad P(x, y) \otimes \models (y, a) \leq \models (x, a).
\]

**Comment.** This definition formulates programming question given above.

**Examples.** Later.
**Enriched Topology and Asymmetry**

**Enriched extents of predicates.** \((L, \leq, \otimes, \bigodot, e)\) unital complete po-ringoid, \((A, \leq, \otimes)\) complete po-groupoid, \((X, P, A, \rhd)\) \(L\)-enriched topological system. Put \(\text{ext}_L : (A, \leq, \otimes) \to (L, \leq, \bigodot)^X\)

\[
\text{ext}_L(a) : X \to L \quad \text{by} \quad \text{ext}_L(a)(x) = \rhd (x, a)
\]

Then \(\text{ext}_L\) preserves arbitrary \(\bigvee\), preserves \(\otimes\) of \(A\) to pointwise lifted \(\otimes\) of \(L\), preserves \(\top^A\) to \(\top_L\). **BUT** ES3 above implies new property for \(\text{ext}_L\):

\[
\forall a \in A, \ \forall x, y \in X, \ P(x, y) \otimes \text{ext}_L(a)(y) \leq \text{ext}_L(a)(x)
\]

Have **extent space** \((X, (\text{ext}_L) \rhd (A))\), where \((\text{ext}_L) \rhd (A) \subseteq L^X\) satisfies: closed under arbitrary \(\bigvee\), closed under \(\otimes\) (lifted pointwise), contains \(\top_L\), and satisfies this **compatibility/enrichment condition**:

\[
\forall u \in (\text{ext}_L) \rhd (A), \ \forall x, y \in X, \ P(x, y) \otimes u(y) \leq u(x)
\]
**Enriched Topology and Asymmetry**

**Enriched Topology** (spaces). Let \((L, \leq, \otimes, \oplus, e)\) be unital complete po-ringoid. Then \((X, P, \tau)\) is \(L\)-enriched/preordered topological space if \((X, P)\) is \((L, \leq, \otimes)\)-preordered set and \(\tau \subset L^X\) such that \(\tau\) is:

- SP1: closed under arbitrary \(\bigvee\)
- SP2: closed under \(\otimes\)
- SP3: contains \(T_L\)
- SP4: \(\forall u \in \tau, \forall x, y \in X, P(x, y) \otimes u(y) \leq u(x)\) (compatibility/enrichment)

**Enriched Topology** (fixed-basis). Let \((L, \leq, \otimes, \oplus, e)\) be unital complete left-residuated po-ringoid. Then \(L\)-\(\text{EnrTop}\) comprises all spaces \((X, P, \tau)\) from above together with all morphisms of the form \(f : (X, P, \tau) \rightarrow (Y, Q, \sigma)\) such that \(f : (X, P) \rightarrow (Y, Q)\) in \(L\)-\(\text{PreSet}\) and \(f : (X, \tau) \rightarrow (Y, \sigma)\) in \(L\)-\(\text{Top}\).

**Theorem.** Let \((L, \leq, \otimes, \oplus, e)\) be unital complete left-residuated po-ringoid. Then \(L\)-\(\text{EnrTop}\) is topological over each of \(L\)-\(\text{PreSet}\) and \(\text{Set}\) w.r.t. the expected forgetful functors.

**Examples.** Later.
Crisp/Many-Valued Specialization Preorders of Many-Valued Spaces

Recall. For \((X, \mathcal{T})\) be topological space: \(x \leq_T y \iff y \in \overline{\{x\}}\). Equivalent to say:

\[ x \leq_T y \iff \forall U \in \mathcal{T}, y \in U \Rightarrow x \in U \]

Defn: \(L\)-Specialization Order. Let \(L\) be complete po-groupoid and \((X, \tau)\) be \(L\)-topological space. Put: \(x \leq_\tau y \iff \forall u \in \tau, u(y) \leq u(x)\). Also dual order.

Defn: \(L\)-Valued Specialization Orders. Let \(L\) be right-residuated complete po-monoid and \((X, \tau)\) be \(L\)-topological space. Put \(P_\tau : X \times X \to L\) by

\[ P_\tau(x, y) = \bigwedge_{u \in \tau} (u(x) \lor u(y)) \]

If \(L\) left-residuated complete po-monoid, then put "dual" \(Q_\tau : X \times X \to L\) by

\[ Q_\tau(x, y) = \bigwedge_{u \in \tau} (u(x) \land u(y)) \]

Note. \(\leq_\tau\) and dual are (crisp) preorders; \(P_\tau\) and \(Q_\tau\) are \(L\)-preorders.

Theorem. Crisp orders induced by \(L\)-valued specialization orders are precisely the crisp \(L\)-specialization orders.
**Enriched Topology and Asymmetry**

**Examples** (spaces to enriched spaces to enriched systems). Let \((L, \leq, \otimes, \ltimes, e)\) be a complete right-residuated po-ringoid such that \((L, \leq, \otimes)\) is a monoid; let \((X, \tau)\) be \((L, \leq, \otimes)\)-topological space.

1. Consider \(L\)-valued specialization order \(P_\tau : X \times X \rightarrow (L, \leq, \otimes)\). Then \(P_\tau\) is compatible with \((X, \tau)\), i.e., \((X, P_\tau, \tau)\) is enriched \(L\)-topological space.
2. Continuing from (1), put \(\models_\tau : X \times \tau \rightarrow L\) by
   \[
   \models_\tau (x, u) = u(x)
   \]
   Then \((X, P_\tau, \tau, \models_\tau)\) is enriched \((L, \leq, \otimes)\)-topological system.
3. "Dualize" (1,2) using \(Q_\tau\) if \(L\) left-residuated equipped with right partial distrib. law, order of pretensorands in compatibility axioms reversed.

**Examples.** Let \((L, \leq, \otimes, \ltimes, e)\) be unital complete po-ringoid, \((A, \leq, \otimes)\) be complete po-groupoid, \((X, P, A, \models)\) be enriched \(L\)-topological system. Then \((X, P, \text{ext}_L^\tau(A))\) is enriched \(L\)-topological space using satisfaction relation from (2) above.
**Examples** (systems to enriched systems). Let \((L, \leq, \otimes, \boxtimes, e)\) be complete right-residuated po-ringoid such that \((L, \leq, \otimes)\) monoid, \((A, \leq, \otimes)\) complete po-groupoid; let \((X, A, \models)\) be \(L\)-topological system. Then \((X, P_{\text{ext}_{L}^{-}}(A), A, \models)\) is enriched \(L\)-topological system. "Dualize" using \(Q_{\text{ext}_{L}^{-}}(A)\) and related concepts and satisfacton relation from (2) above.

**Examples** (spectra to enriched systems/spaces). Let \((L, \leq, \otimes, \boxtimes, e)\) be complete right-residuated po-ringoid such that \((L, \leq, \otimes)\) monoid, \((A, \leq, \otimes)\) complete po-groupoid. Put:

\[
Lpt(A) = \{p : A \to L \mid p \text{ preserves } \bigvee, \otimes \text{ to } \boxtimes, \top\},
\]

\[
P_{A} : Lpt(A) \times Lpt(A) \to L \text{ by } P_{A}(p, q) = \bigwedge_{a \in A} (p(a) \lor q(a)),
\]

\[
\models_{A} : Lpt(A) \times A \to L \text{ by } \models_{A}(p, a) = p(a)
\]

Then \((Lpt(A), P_{A}, A, \models_{A})\) is enriched \(L\)-topological system. For corresponding enriched \(L\)-topological extent space \((Lpt(A), P_{A}, \text{ext}_{L}^{-}(A))\), \(P_{A} = P_{\text{ext}_{L}^{-}(A)}\). "Dualize" using \(Q_{A}, Q_{\text{ext}_{L}^{-}}(A)\) and related concepts.
**Examples** ((bit)strings from alphabets). Let:

\[ \Sigma \text{ be set with } |\Sigma| \geq 2, \text{ viewed as "alphabet"} \]

\[ \Sigma^{*\omega} =: \{ \text{countable strings of letters from } \Sigma \} \]

\[ \mathcal{B} =: 2^\omega = \{ \text{countably infinite strings of letters from } 2 \} \text{—complete Bool. alg. with } \otimes = \varnothing = \wedge \]

Put \( P : \Sigma^{*\omega} \times \Sigma^{*\omega} \to \mathcal{B} \) by—for \( n \in \mathbb{N} \)—

\[
P(\sigma_1, \sigma_2)(n) = \begin{cases} 1, & \text{if } \sigma_1(n), \sigma_2(n) \text{ exist and } \sigma_1(n) = \sigma_2(n) \\ 0, & \text{otherwise} \end{cases}
\]

Each \( P(\sigma_1, \sigma_2) \) is *comparison bitstring*. It follows that \( (\Sigma^{*\omega}, P) \) is \( \mathcal{B} \)-preordered set.

Now for \( \alpha \in \Sigma \), put \( p^\alpha : \Sigma^{*\omega} \to \mathcal{B} \) by

\[
p^\alpha(\sigma)(n) = \begin{cases} 1, & \text{if } \sigma(n) \text{ exists and } \sigma(n) = \alpha \\ 0, & \text{otherwise} \end{cases}
\]

Each \( p^\alpha(\sigma) \) is *indicator string* and member of \( \mathcal{B}^{\Sigma^{*\omega}} \). Now put
\[ Q = \langle \langle \{ p^a : a \in \Sigma \} \rangle \rangle \subset B^{\Sigma^{*\omega}}, \]

the \( B \)-topology having subbasis the indicator strings. It can be shown that \( Q \) is compatible with \( P \); hence \((\Sigma^{*\omega}, P, Q)\) is \( B \)-enriched topological space.

Finally, \((\Sigma^{*\omega}, P, Q, \models_Q)\) is \( B \)-enriched topological system, where \( \models_Q : \mathbb{X} \times Q \to L \) by

\[ \models_Q (x, u) = u(x) \]

"String" spaces encountered again below w.r.t. many-valued \( T_1 \) separation.
**Antisymmetry & $L-T_0$ Many-Valued Topological Spaces**

**Defn** (antisymmetry and partial orders). Let $L$ be a unital po-groupoid. Then $P : X \times X \rightarrow L$ on $X$ is ($L$-)antisymmetric if

$$\forall x, y \in X, \ P(x, y) \geq e, \ P(y, x) \geq e \Rightarrow x = y;$$

and $L$-partial order is antisymmetric $L$-preorder; and $L$-Poset is full subcategory of $L$-PreSet comprising all $L$-posets.

**Comment.** Above definition justified in several directions: skeleton of each $L$-PreSet; quotients of $L$-presets (construction requires $L$-antisymmetry, IIA operators/involutions, and $L$-symmetry—latter below with $L-T_1$ issues); furnishes right-adjoint of each $L$-PreSet; generalizes classical result that Poset is monotopological construct; characterizes fundamental $L-T_0$ axiom in many-valued topology.

**Theorem.** For each complete unital po-groupoid $L$, $L$-Poset is monotopological over Set w.r.t. expected forgetful functor; so Poset is monotopological construct.
**Enriched Topology and Asymmetry**

**Defn** ($T_0$ separation). Let $L$ be complete po-groupoid and $(X, \tau)$ be $L$-topological space. Then $(X, \tau)$ is $L-T_0$ if

$$\forall x, y \in X, \ x \neq y \Rightarrow \exists u \in \tau, \ u(x) \neq u(y)$$

The $L-T_0$ axiom has different formulations (e.g., injectivity of the $L$-Stone second comparison maps), and is well-established via representation theorems and compactification reflectors, including two successful forms of sobriety. Examples include spectra and the fuzzy real lines and unit intervals. More justification below.
**Main Theorem.** Assume: \( L \) complete po-groupoid, \( L \)-topological space \((X, \tau)\).

1. \( \leq_\tau \) (and dual) is antisymmetric (and po) iff \((X, \tau)\) is \(L-T_0\).

2. Further assume \( L \) right-residuated complete po-monoid. Then
\[
P_\tau : X \times X \to L \text{ is antisymmetric (and po) iff } (X, \tau) \text{ is } L-T_0.
\]

3. Further assume \( L \) left-residuated complete po-monoid. Then
\[
Q_\tau : X \times X \to L \text{ is antisymmetric (and po) iff } (X, \tau) \text{ is } L-T_0.
\]

4. For \( L \) a right[left]-residuated complete po-monoid, \( \leq_\tau \) (and dual) is antisymmetric iff \( P_\tau [Q_\tau] \) is antisymmetric. Hence for \( L \) unital quantale, \( \leq_\tau \) (and dual) is antisymmetric iff \( P_\tau \) is antisymmetric iff \( Q_\tau \) is antisymmetric.

5. For \( L \) DeMorgan frame with antitone involution \( ' : L \to L \), the \( L \)-valued hemimetric \( P_\tau' : X \times X \to L \) induced by \( P_\tau \) satisfies the following positive definiteness condition—
\[
[\forall x, y \in X, P_\tau'(x, y) = P_\tau'(y, x) = \bot \iff x = y]
\]
if and only if \((X, \tau)\) is \(L-T_0\).
Notes.
(1) $L$-antisymmetry essentially "same as" traditional antisymmetry, so appropriate generalization.

(2) To check $L$-antisymmetry of $P_\tau$, suffices to check antisymmetry of $\leq_\tau$. But latter always included in former (for right-residuated complete po-monoid case) since $\chi^e_{\leq_\tau} \leq P_\tau$, where $\chi^e_{\leq_\tau} \equiv \chi_{\leq_\tau} \land e : X \times X \to \{\bot, e\} \subset L$. 
Symmetry & $L-T_1(1) / L-T_1(2)$ Separation in Many-Valued Topology

Recall. For $L$ unital groupoid, $P$ is $L$-valued preorder on $X$ if:

- **P0**: $P : X \times X \to L$ is mapping (degrees of precedence)
- **P1**: $\forall x \in X, \ e \leq P(x, x)$ (reflexivity)
- **P2**: $\forall x, y, z \in X, \ P(x, y) \otimes P(y, z) \leq P(x, z)$ (transitivity)

Can also consider:

- **P3**: $\forall x, y \in X, \ P(x, y) \geq e, \ P(y, x) \geq e \Rightarrow x = y$ (antisymmetry)

**Defn** (symmetry). Let $X$ be set and $(L, \leq, \otimes, e, *)$ be unital IIA po-groupoid. Then $P$ is a symmetric $L$-valued relation on $X$ if $P$ satisfies P0 and

- **P4**: $\forall x, y \in X, \ P(x, y) = P^*(y, x)$ (symmetry),

* $: L \to L$ is IIA oper./invol. ($a^{**} = a, \ a \leq b \Rightarrow a^* \leq b^*, \ (a \otimes b)^* = b^* \otimes a^*$).

**Remark.** If $\otimes$ is commutative and $*$ is chosen as $id_L$, then P4 becomes

$\forall x, y \in X, \ P(x, y) = P(y, x)$
**Note.** Many-valued symmetry above central to many-valued antisymmetry capturing quotients of $L$-presets as $L$-posets and to the latter comprising the right adjoint of the former. Important justification of many-valued symmetry.

**Strategy.** Know antisymmetry characterizes $T_0$ for traditional topology, symmetry characterizes $T_1$ for traditional topology, and antisymmetry (crisp or many-valued) characterizes $L-T_0$ for many-valued topology. Propose to define $L-T_1$ for many-valued topology using above many-valued symmetry. Accordingly, propose to define *asymmetry* for many-valued topology as the denial of above many-valued symmetry.

**Defn.** For $L$ a right-residuated complete po-monoid, an $L$-topological space $(X, \tau)$ or its topology $\tau$ is *asymmetric* if $\leq_\tau$ or $P_\tau$ is not symmetric.

Typically, asymmetric space is also $L-T_0$.

**Standing Assumption.** Until stated otherwise, $L$ in sequel is unital IIA quantale.
**ENRICHED TOPOLOGY AND ASYMMETRY**

IIA induced spaces. Let $(X, \tau)$ be $L$-topological space. Put:

$$\tau^* = \{ u^* : u \in \tau \}, \quad T = \tau \lor \tau^*$$

Then $(X, \tau^*), (X, T)$ are $L$-topological spaces; and $T$ is smallest $*$-invariant topology containing $\tau$ (so $T^* = T$).

**Example.** Suppose $\otimes$ non-commutative with $* \neq id_L$; $\exists a \in L, a^* \neq a$. Put

$$\tau = \{ \bot, a, \top \}$$

for some set $X$. Then $(X, \tau)$ is $L$-topological space and $\tau \not\subseteq T$.

**Lemma.** Let $X$ be set and $x, y \in X$.

1. $P_{\tau}^* : X \times X \to L$ by $P_{\tau}^*(x, y) = \bigwedge_{u \in \tau} (u^*(y) \imps u^*(x))$.

2. $P_{\tau}^*(x, y) = Q_{\tau^*}(y, x)$.

3. $P_{\tau}$ symmetric iff $P_{\tau} = Q_{\tau^*}$; $Q_{\tau}$ symmetric iff $Q_{\tau} = P_{\tau^*}$; $P_{\tau}$ symmetric iff $Q_{\tau}$ symmetric.

4. $P_T$ symmetric iff $P_T = Q_T$ iff $Q_T$ symmetric.
**Theorem** *(L-specialization vis-a-vis L-valued specialization w.r.t. symmetry).*

(1) Let $P_\tau$ be symmetric. Then:
- (a) $\leq_\tau$ symmetric;
- (b) $\leq_{\tau^*}$ coincides with $\leq_\tau$ and hence symmetric;
- (c) $\leq_T$ coincides with $\leq_\tau$ and hence symmetric.

(2) Converse to (1)(a) fails, even when $\leq_\tau$ additionally assumed antisymmetric, i.e., $(X, \tau)$ also $L$-$T_0$.

**Examples.**

(1) Let $X = \{x, y\}, L = \{\bot, a, b, \top\}: \otimes = \land, \vee = \lor = \rightarrow, * = id_L$, so $P^*_\tau(y, x) = P_\tau(y, x)$. Put: $u(x) = \bot, u(y) = a; v(x) = b, v(y) = \top; o(x) = b, o(y) = a; \tau = \{\bot, u, v, o, \top\}$. Then $(X, \tau)$ is $L$-topological space which is $L$-$T_0$ and for which $\leq_\tau$ is both antisymmetric and symmetric (and hence trivial), and $P_\tau$ is $L$-antisymmetric but not $L$-symmetric.

(2) As in (1), but not include $o$. Then $(X, \tau)$ is $L$-topological space which is $L$-$T_0$ and for which $\leq_\tau$ is antisymmetric but not symmetric, and $P_\tau$ is $L$-antisymmetric but not $L$-symmetric.
**Defn** (*L*-T₁ separation axioms). Let \((X, \tau)\) be *L*-topological space.

1. Suppose *L* semiquantale. \((X, \tau)\) is *L-*T₁ in *first sense*, or *L-*T₁(1), if \(\leq_\tau\) antisymmetric and symmetric.

2. Suppose *L* unital IIA quantale. \((X, \tau)\) is *L-*T₁ in *second sense*, or *L-*T₁(2), if \(P_\tau\) antisymmetric and symmetric. So, \((X, \tau)\) is *L-*T₁(2) if \(P_\tau\) satisfies:
   
   P₁: \(\forall x \in X,\ P_\tau(x, x) \geq e;\)
   
   P₂: \(\forall x, y, z \in X,\ P_\tau(x, y) \otimes P_\tau(y, z) \leq P_\tau(x, z);\)
   
   P₃: \(\forall x, y \in X,\ P_\tau(x, y) \geq e,\ P_\tau(y, x) \geq e \implies x = y;\)
   
   P₄: \(\forall x, y \in X,\ P(x, y) = P^*(y, x).\)

**Corollary.** Let *L* be unital IIA quantale and \((X, \tau)\) be *L*-topological space.

1. *L-*T₁(1) implies *L-*T₀, but not conversely.
2. *L-*T₁(2) implies *L-*T₀, but not conversely.
3. *L-*T₁(2) implies *L-*T₁(1), but not conversely.

**Comment.** When evaluating an *L*-topological space for asymmetry, it merely suffices to determine if \(\leq_\tau\) is not symmetric; and if the space is *L-*T₀, then it suffices to determine that the space is not *L-*T₁(1).
Reconciliation with Kubiak (1995) $L-T_1$ axiom. Let $L$ be semiquantale and $(X, \tau)$ be $L$-topological space.

(1) $(X, \tau)$ is $L-T_1(K)$ if:

$$\forall x, y \in X, x \neq y \Rightarrow \exists u, v \in \tau, u(y) \ngeq u(x) \text{ and } v(x) \ngeq v(y)$$

(2) Rewrite $L-T_1(K)$:

$$\forall x, y \in X, x \neq y \Rightarrow x \triangleleft \tau y \text{ and } y \triangleleft \tau x. \quad (K1)$$

(3) Rewrite $L-T_1(1)$:

$$\forall x, y \in X, x \triangleleft \tau y \Leftrightarrow y \triangleleft \tau x. \quad (K2)$$

(4) Assume $L-T_1(K)$, suppose $x \triangleleft \tau y$. Then (K1) gives $x = y$, reflexivity of $\leq_\tau$ gives $y \leq_\tau x$. So $L-T_1(1)$ holds.

(5) Assume $L-T_1(1)$, suppose $x \neq y$. Antisymmetry of $\leq_\tau$ yields $x \triangleleft \tau y$ or $y \triangleleft \tau x$. But (K2) in each case yields $x \triangleleft \tau y$ and $y \triangleleft \tau x$. By (K1), $L-T_1(K)$ holds.

(6) So $L-T_1(1) \Leftrightarrow L-T_1(K)$. Different motivations: $L-T_1(K)$ motivated by giving symmetric version of $L-T_0$; $L-T_1(1)$ motivated by symmetry of $L$-specialization order $\leq_\tau$. 
Spectra Related Examples. Let \(\text{SQuant}\) be category of all semiquantales and all morphisms being those maps preserving arbitrary \(\bigvee\) and \(\otimes\); and let \(\text{Squant}_\top\) be subcategory of all semiquantales and those semiquantale morphisms which also present \(\top\).

Fix semiquantale \(L\), and let \(A\) be any semiquantale. Put:

\[
Lpt(A) = \text{SQuant}_\top(A, L) = \{p : A \to L \mid p \text{ preserves } \bigvee, \otimes, \top\},
\]

\[
\Phi_L : A \to L^{Lpt(A)} \text{ by } \Phi_L(a) : Lpt(A) \to L \text{ by } \Phi_L(a)(p) = p(a).
\]

Then \(Lpt(A) = (Lpt(A), (\Phi_L)^\sim(A))\) is \(L\)-topological space, the \(L\)-spectrum of \(A\).

Defn. Let \(A\) be semiquantale.

(1) \(c \in A - \{\top\}\) is \(\otimes\)-prime if: \(\forall a, b \in A, a \otimes b \leq c \iff a \leq c \text{ or } b \leq c\). \(\Pr_{\otimes}(A)\) set of all \(\otimes\)-primes of \(A\).

(2) \(A\) has two related \((\otimes-)\)primes if \(\exists a, b \in \Pr_{\otimes}(A), a \leq b\) and \(a \neq b\).

Lemma. \(Lpt(A)\) is \(L-T_1(1)\) iff \(\forall p, q \in Lpt(A), q \leq p \iff p \leq q\) (in \(Lpt(A)\)).

Theorem. Assume \(L\) integral, \(\bot\) annihilator for \(\otimes\) in \(L, A\). Then \(L\) consistent & \(A\) has two related primes \(\Rightarrow Lpt(A)\) not \(L-T_1(1)\). "Iff" if \(L = 2\).
Enriched Topology and Asymmetry

**Example.** Assume $L$ integral & consistent, $\perp$ annihilator for $\otimes$ in $L, A$, and $\mathcal{J}_{\text{cof}}$ cofinite topology of $\mathbb{R}$. Then $L Pt(\mathcal{J}_{\text{cof}})$ not $L-T_1(1)$.

**Examples.** Let $L$ be consistent, complete DeMorgan algebra; let $\mathbb{R}(L), \mathbb{R}_l(L), \mathbb{R}_r(L), \mathbb{I}(L), \mathbb{I}_l(L), \mathbb{I}_r(L)$ be $L$-fuzzy real line, $L$-fuzzy left-handed real line, $L$-fuzzy right-handed real line, $L$-fuzzy unit interval, $L$-fuzzy left-handed unit interval, $L$-fuzzy right-handed unit interval.

1. All are $L-T_0$.
2. $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are $L-T_1(1)$; the others are not $L-T_1(1)$.
3. Recall $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$, so-called "alternative" fuzzy real line and unit interval, are defined to be $L Pt(\mathcal{J})$, where $\mathcal{J}$ is standard topology on $\mathbb{R}$ and $\mathbb{I}$, respectively. For $L$ a complete Boolean algebra, $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$ are both $L$-sober and $L-T_1(1)$.
4. For $L$ a complete Boolean algebra, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are both $L$-sober and $L-T_1(1)$. (This follows from (3)).

**Examples.** Recall $\mathcal{B}$-enriched topological spaces $(\Sigma^{*\omega}, P, Q)$ described above.

1. $P$ is reflexive, transitive, compatible, antisymmetric, symmetric.
2. $\mathcal{B}$-valued specialization order $P_Q = P$. Hence $(\Sigma^{*\omega}, P, Q)$ is $L-T_1(2)$. 
Examples (behavior w.r.t. $M_L \rightarrow G_\chi$). For $L-T_0$ and $L-T_1(1)$ issues, $L$ is semiquantale; for $L-T_1(2)$ issues, $L$ is integral IIA quantale.

(1) $(X, T) \ T_0$ iff $G_\chi(X, T) \ L-T_0$; $(X, T) \ T_1$ iff $G_\chi(X, T) \ L-T_1(1)$.
(2) $M_L$ reflects $T_0 [T_1]$ to $L-T_0 [L-T_1(1)]$.
(3) $G_\chi$ preserves $T_1$ to $L-T_1(2)$, and reflects $L-T_1(2)$ to $T_1$. Hence $G_\chi$ preserves asymmetry.
(4) $M_L$ reflects $T_1$ to $L-T_1(2)$, and hence preserves $L$-asymmetry.

Examples (behavior w.r.t. $\omega_L \rightarrow \iota_L$). Generally, $L$ is semiquantale.
(1) Let $(X, \tau) \in |L-\text{Top}|$. Then $(X, \tau)$ is $L-T_1(1)$ iff $\iota_L(X, \tau)$ is $T_1$. Hence $\iota_L$ reflects and preserves $L$-asymmetry.
(2) Let $(X, \mathcal{T}) \in |\text{Top}|$. Then $(X, \mathcal{T})$ is $T_1$ implies $\omega_L(X, \mathcal{T})$ is $L-T_1(1)$, in which case $\omega_L$ reflects $L$-asymmetry; and the converse holds if $L$ is completely distributive ($\otimes = \wedge$), in which case $\omega_L$ preserves asymmetry.
(3) Continuing (2), if $L$ admits an endomorphism other than the identity, then for $(X, \mathcal{T})$ being $T_1$, $\omega_L(X, \mathcal{T})$ is both $L-T_1(1)$ and not $L$-sober.
Other Separation Issues.

(1) Strong $L-T_2$ (Höhle) implies weak $L-T_2$ (Kubiak) implies $L-T_1(1)$. But neither Hausdorff axiom implies $L$-sober: for any $L$ complete DeMorgan frame which is non Boolean, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are strong $L-T_2$ but not $L$-sober.

(2) For $L$ semiquantale, have: $\tau_L$-Hausdorff implies $L-T_1(1)$ implies $L-T_0$, and $\tau_L$-Hausdorff implies $\tau_L$-sober implies $L-T_0$, and $L-T_1(1)$ and $\tau_L$-sober unrelated. For $L$ completely distributive DeMorgan algebra, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are $\tau_L$-Hausdorff and $\tau_L$-sober and $L-T_1(1)$; but if $L$ is also non-Boolean, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are not $L$-sober. So $\tau_L$-Hausdorff $\nRightarrow$ $L$-sobriety.
**Example** (non-commutative conjunctions in programming / web searches).

Two numerical variables $x$, $y$ related to website.

(1) $y$ counts how many times website has been visited.
(2) Whenever $y$ is used in expression, website is accessed before value in $y$ is read, so $y$ always updated.
(3) Each time $y$ is read, its value increases by 1.
(4) Predicates related to website take on traditional truth values $T$ or $F$. The conjunction of predicates $\varphi$, $\psi$ written $\varphi \land \psi$, which is $T$ if and only if each of $\varphi$, $\psi$ is $T$.
(5) Website not accessed when conjunctions of associated predicates are formed.
(6) Conjunction $\varphi \land \psi$ read left-to-right, first $\varphi$, then $\psi$.
(7) Assume:

   current value of $x$ is 9, current value of $y$ is 8

   Consider predicates $P : [x = y]$, $Q : [y \geq 10]$. 
(8) Truth value of $P \sqcap Q$ is $T$: reading $P$ first updates $y$ to $y = 9$; reading $Q$ second updates $y$ yet again, giving $y = 10$; so each of $P, Q$ is $T$.

(9) Truth value of $Q \sqcap P$ is $F$: reading $Q$ first updates $y$ to $y = 9$; reading $P$ second updates $y$ yet again, giving $y = 10$; so each of $P, Q$ is $F$.

(10) Note also that $P \sqcup Q$ is $T$, while $Q \sqcup P$ is $F$.

(11) Note if $Q$ is replaced by $\hat{Q}$: $[y \geq 9]$, then:

$P \sqcap \hat{Q}$ is $T$, \hspace{1cm} $\hat{Q} \sqcap P$ is $F$

$P \sqcup \hat{Q}$ is $T$, \hspace{1cm} $\hat{Q} \sqcup P$ is $T$
References


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