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Enriched Topology and Asymmetry

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Motivations for Asymmetry

First Motivation: Quasimetric Spaces

Consider possible conditions for (X, d), where $d : X \times X \to \mathbb{R}$:

(M1) $\forall x, y \in X,\ d(x, y) \geq 0$ (non-negativity)
(M2) $\forall x, y \in X,\ [d(x, y) > 0 \text{ or } d(y, x) > 0] \Leftrightarrow x \neq y$ (weak pos. definiteness)
(M2a) $\forall x, y \in X,\ d(x, y) = 0 \text{ if } x = y$ (zero-distance on diagonal)
(M2b) $\forall x, y \in X,\ [d(x, y) > 0 \text{ and } d(y, x) > 0] \Leftrightarrow x \neq y$ (strong pos. def.)
(M3) $\forall x, y \in X,\ d(x, y) = d(y, x)$ (symmetry)
(M4) $\forall x, y, z \in X,\ d(x, z) \leq d(x, y) + d(y, z)$

Note: M2b $\Rightarrow$ M2, M3 $\Rightarrow$ (M2 $\Leftrightarrow$ M2b), M1 $\Rightarrow$ (M2 $\Rightarrow$ M2a, M2b $\Rightarrow$ M2a),
M2a $\Leftrightarrow s(x, y) = 1 - d(x, y)$ reflexive

Defn. (X, d) is: *metric space* if M1,M2,M3,M4;
pseudometric space if M1,M2a,M3,M4
quasimetric space if M1,M2b,M4
hemimetric space if M1,M2a,M4

For hemimetric space (X, d), have topology $\mathfrak{T}_d$ generated by ε-balls. Then for hemimetric d, $(X, \mathfrak{T}_d)$ is: $T_0 \Leftrightarrow$ M2; $T_1 \Leftrightarrow$ M2b; T_2 if M2 and M3.

Second Motivation: Specialization preorders. Let $(X, \mathfrak{T})$ be topological space; put

$$x \leq_{\mathfrak{T}} y \;\Leftrightarrow\; y \in \overline{\{x\}}$$

This (or dual) is *specialization* preorder. It is antisymmetric iff $(X, \mathfrak{T})$ is T_0; it is antisymmetric and symmetric (and hence equality $=$) iff $(X, \mathfrak{T})$ is T_1.

Working with sober, non-T_1 spaces means working with asymmetric specialization preorders.

Defn. Topological space $(X, \mathfrak{T})$ *asymmetric* if specialization order not symmetric.

In practice, asymmetry means T_0 but not T_1 spaces.

Order-Theoretics: PO-Groupoids, IIA Operators/Involutions

Groupoid/magma: $(X,\otimes)$, $\otimes: X\times X\to X$ ("flat") *semitensor*

unital: $(X,\otimes,e)$, two-sided identity e
monoid: $(X,\otimes,e)$ unital semigroup
semiquantale: $(L,\le,\otimes)$ with $(L,\le)$ complete lattice, $(L,\otimes)$ groupoid

Po-groupoid: $(L,\le,\otimes)$ with $(L,\le)$ poset, $(L,\otimes)$ groupoid, $\otimes$ isotone both var.

IIA (or involutive) po-groupoid: $(L,\le,\otimes,{}^*)$, $(L,\le,\otimes)$ po-groupoid, ${}^*: L\to L$
involutive, isotone, anti-automorphism in these senses:
involutive ($a^{**}=a$), isotone ($a\le b \Rightarrow a^*\le b^*$),
interchanges with $\otimes$ ($(a\otimes b)^*=b^*\otimes a^*$)
left[right]-residuated po-groupoid: $(L,\le,\otimes)$ with $\searrow$ $[\swarrow]: L\times L\to L$ s.t.

$$a\searrow b\ge c \Leftrightarrow a\otimes c\le b \quad [b\swarrow a\ge c \Leftrightarrow c\otimes a\le b]$$

po-monoid: $(L,\le,\otimes)$ unital po-groupoid with $\otimes$ associative *complete po-groupoid*: ordered semiquantale, i.e., $(L, \leq)$ complete lattice

semiframe/semilocale: $\otimes = \wedge$ (binary), integral

complete po-semigroup: $\otimes$ associative

quantale: $\otimes$ distributes on both sides across arbitrary $\bigvee$, $\otimes$ assoc.

frame/locale: $\otimes = \wedge$ (binary), integral

Semiquantale morphisms: preserve $\bigvee, \otimes$

Comment. Given $(L, \leq, \otimes)$, an IIA-operator/involution may be chosen to be $* = id_L$ iff $\otimes$ is commutative. Allowable choices of IIA-operators/involutions for $\otimes$ roughly gauge deviation of $\otimes$ from being commutative. Non-commutative $\otimes$ and associated IIA-operators/involutions $*$ tied to possible symmetries of L-valued specialization orders for L-topological spaces.

Example. $(L, \leq, ')$ DeMorgan algebra (complete). Construct $(S(L), \leq, \circ, ^*)$ by:

$$S(L) = \{f : L \to L \mid f \text{ preserves } \bigvee\},$$
$$\leq \text{ taken pointwise},$$
$$\circ \text{ composition of functions},$$
$$f^*(a) = (f^{\vdash}(a'))',$$

where $f^{\vdash} : L \to L$ is right adjoint of f guaranteed by AFT($\bigvee$) and given by

$$f^{\vdash}(b) = \bigvee_{f(a) \leq b} a.$$

Then $(S(L), \leq, \circ, ^*)$ is unital, IIA quantale; and it is both non-integral and non-commutative iff $|L| \geq 3$.

Powerset monads and topology. Fix X, Y sets, L semiquantale, $f : X \to Y$; have $f_L^{\to} : L^X \to L^Y$, $f_L^{\leftarrow} : L^X \leftarrow L^Y$, $f_{L\to} : L^X \to L^Y$, given respectively by

$$f_L^{\to}(a)(y) = \bigvee_{f(x)=y} a(x),\ f_L^{\leftarrow}(b) = b \circ f,\ f_{L\to}(a) = \bigwedge_{a \leq f_L^{\leftarrow}(b)} b.$$

Fact $f_L^{\rightarrow} \dashv f_L^{\leftarrow} \dashv f_{L\rightarrow}$

(X, τ) is *L-topological space* if $\tau \subset L^X$ closed under $\bigvee, \otimes, \underline{\top}$. $f : (X, \tau) \to (Y, \sigma)$ *L-continuous* if $\forall v \in \sigma, f_L^{\leftarrow}(v) \in \tau$.

Fact L-**Top** topological construct.

Sets Enriched by PO-Monoids

Enriched category $\mathcal{C}$ over a monoidal category $(\mathcal{M}, \otimes, I, \alpha, \lambda, \rho)$ is class of objects with data C0, C1, and C2 subject to axioms D1, D2, and D3:

C0: $\forall a, b \in \mathcal{C}, \exists! \mathcal{C}(a,b) \in |\mathcal{M}|$ (existence of hom-objects)
C1: $\forall a \in \mathcal{C}, \exists\, id_a : I \to \mathcal{C}(a,a)$ (existence of identities)
C2: $\forall a, b, c \in \mathcal{C}, \exists\, !\, \circ_{abc} : \mathcal{C}(b,c) \otimes \mathcal{C}(a,b) \to \mathcal{C}(a,c)$ (comp. of hom-objects)

D1: $\forall a, b, c, d \in \mathcal{C},\ (\circ_{abd}) \circ (\circ_{bcd} \otimes \mathbf{1}_{\mathcal{C}(a,b)}) = (\circ_{acd}) \circ (\mathbf{1}_{\mathcal{C}(c,d)} \otimes \circ_{abc}) \circ \alpha$

D2: $\forall a, b \in \mathcal{C},\ \lambda = (\circ_{abb}) \circ (id_b \otimes \mathbf{1}_{\mathcal{C}(a,b)})$

D3: $\forall a, b \in \mathcal{C},\ \rho = (\circ_{aab}) \circ (\mathbf{1}_{\mathcal{C}(a,b)} \otimes id_a)$

Comment. Po-monoid $(L, \leq \otimes)$, taken as a preordered category, is a monoidal category in which $\otimes$ is the categorical tensor product, *I* is the unit *e*, and the associator α and the unitors λ, ρ are all identities..

Prop. If $(L, \leq \otimes)$ is po-monoid, then set X replacing $\mathcal{C}$ is L-enriched category iff there is *enrichment relation*, or *L-(valued) preorder P* on X s.t.:

E0: $P : X \times X \to L$ is a mapping (*degrees of comparison/precedence*)
E1: $\forall x \in X,\ e \leq P(x,x)$ (*reflexivity*)
E2: $\forall x, y, z \in X,\ P(y,z) \otimes P(x,y) \leq P(x,z)$ (*transitivity*)

Defn. For $(L, \leq \otimes)$ a unital po-groupoid, (X, P) is an *L-preordered set*, or (X, L, P) is a *preordered set*, if:

P0: $P : X \times X \to L$ is a mapping (*degrees of comparison/precedence*)
P1: $\forall x \in X,\ e \leq P(x,x)$ (*reflexivity*)
P2: $\forall x, y, z \in X,\ P(x,y) \otimes P(y,z) \leq P(x,z)$ (*transitivity*)

Defn. Given $\mathcal{M}$-enriched categories $\mathcal{C}$, $\mathcal{D}$, $F : \mathcal{C} \to \mathcal{D}$ is *$\mathcal{M}$-enriched functor* if:

F1: $\forall a \in \mathcal{C},\ \exists!\ F(a) \in \mathcal{D}$
F2: $\forall a, b \in \mathcal{C},\ \exists!\ F_{ab} \in \mathcal{M}(\mathcal{C}(a,b), \mathcal{D}(F(a), F(b)))$
F3: $\forall a \in \mathcal{C},\ F_{aa} \circ id_a = id_{F(a)}$ (in $\mathcal{M}$)
F4: $\forall a, b, c \in \mathcal{C},\ F_{ac} \circ (\circ_{abc}) = (\circ_{F(a)F(b)F(c)}) \circ (F_{bc} \otimes F_{ab})$ (in $\mathcal{M}$)

Prop. Let L be po-monoid. Given L-preordered sets (X, P), (Y, Q) taken as L-enriched categories, then $f : (X, P) \to (Y, Q)$ is L-enriched functor iff $f : X \to Y$ is mapping such that

$$P(x, y) \le Q(f(x), f(y)).$$

Defn. For $(L, \le \otimes)$ a unital po-groupoid, an *L-isotone map* $f : (X, P) \to (Y, Q)$ is a mapping $f : X \to Y$ such that $P(x, y) \le Q(f(x), f(y))$.

Defn (fixed-basis). For unital po-groupoid L, the category L-**PreSet** comprises L-preordered sets and L-isotone mappings together with the composition and identities of **Set**.

Theorem (fixed-basis). For each complete unital po-groupoid, the category L-**PreSet** is a topological category over **Set** w.r.t. expected forgetful functor; i.e., each L-**PreSet** is a topological construct. Hence so is **PreSet**.

Comment. Variable-basis enriched functors can be defined for enriched category theory, with corresponding variable-basis morphisms between preordered sets (X, L, P). Topologicity quite delicate; see DMR FSS 2014.

Many-Valued Enriched Topological Systems & Their Extent Spaces

Question. Suppose bitstring *x* precedes bitstring *y* to some degree α, and bitstring *y* satisfies predicate *a* to some degree β. How should bitstring *x* satisfy predicate *a* to at least some degree related to both α and β? What if conjunction of predicates is non-commutative? Potential applications in data-mining and pattern-matching.

Complete po-groupoid based topological systems. Let *A*, *L* be complete po-groupoids. An *L-topological system* $(X, A, \vDash)$ comprises: set *X*, and many-valued *satisfaction relation* $\vDash\ : X \times A \to L$ which respectively satisfies *join, pretensor, top interchange laws*:

$$\forall x \in X,\ \forall \{a_\gamma\}_{\gamma\in\Gamma} \subset A,\ \ \vDash\left(x, \bigvee_{\gamma\in\Gamma} a_\gamma\right) = \bigvee_{\gamma\in\Gamma} \vDash(x, a_\gamma),$$

$$\forall x \in X,\ \forall a, b \in A,\ \ \vDash(x, a \otimes b) = \ \vDash(x, a) \otimes \ \vDash(x, b),$$

$$\forall x \in X,\ \ \vDash(x, \top) = \top.$$

Note. $\forall x \in X,\ \ \vDash(x, \bot) = \bot$; i.e., each bitstring never satisfies **false**.

Extents of predicates. Let $(X, A, \vDash)$ be L-topological system. The *L-extent* operator $ext_L : A \to L^X$ given by

$$ext_L(a)(x) = \vDash(x, a)$$

Then ext_L preserves arbitrary $\bigvee$, $\otimes$, $\top$ (to $\underline{\top}$) from A to L^X. So *extent space*

$$(X, (ext_L)^{\to}(A))$$

is L-topological space as defined above.

Notes.

(1) *Each* object of L-**Top** is produced in this way, i.e., is extent space.

(2) Information (Chu) systems named by the character of their extent spaces; above justifies name "L-topological systems".

PO-Ringoids. $(L,\le,\otimes,\circledast)$ is *po-ringoid* means:

R1: $(L,\le,\otimes)$ po-groupoid with pretensor $\otimes$
R2: $(L,\le,\circledast)$ po-groupoid with multiplication $\circledast$
R3: *left-partial distributive law* of $\otimes$ over $\circledast$ holds:

$$a\otimes(b\circledast c)\le(a\otimes b)\circledast(a\otimes c)$$

Idea. Want to add topological system structure or topological structure to preset; $(L,\le,\otimes)$ with completeness and unit undergirds order, $(L,\le,\circledast)$ with completeness undergirds topology; left-partial distributive law ties together.

Note. May add conditions to po-ringoids. *Groupoidal quantale*: unital complete po-ringoid $(L,\le,\otimes,\circledast,e)$ with $(L,\le,\otimes,e)$ unital quantale and $\otimes$ distributes partially over $\circledast$ from both left and right.

Some Examples.
(1) Complete lattice $(L,\le,\vee,\wedge,\perp)$ unitial complete po-ringoid; if also distributive, then $(L,\le,\wedge,\vee,\top)$ integral complete po-ringoid.
(2) $([0,1],\le,t_{Ł},\wedge,1)$ integral groupoidal quantale.
(3) $(S(L),\le,\circ,\circledast_1,id_L),(S(L),\le,\circ,\circledast_2,id_L)$ non-integ. groupoidal quantales[note].

Enriched topological systems. Let $(L, \leq, \otimes, \circledast, e)$ be a unital complete po-ringoid and $(A, \leq, \otimes)$ be a complete po-groupoid. An *L-enriched topological system* $(X, P, A, \vDash)$ comprises the following:

ES1: L-preordered set (X, P) using unital po-groupoid $(L, \leq, \otimes)$

ES2: L-topological system $(X, A, \vDash)$, where satisfaction relation $\vDash\ : X \times A \to L$ respectively satisfies join, *pretensor-multiplication*, top interchange laws using complete po-groupoids $(L, \leq, \circledast)$ and $(A, \leq, \otimes)$—

$$\forall x \in X,\ \forall \{a_\gamma\}_{\gamma \in \Gamma} \subset A,\ \ \vDash\left(x, \bigvee_{\gamma \in \Gamma} a_\gamma\right) = \bigvee_{\gamma \in \Gamma} \vDash (x, a_\gamma),$$

$$\forall x \in X,\ \forall a, b \in A,\ \ \vDash (x, a \otimes b) = \ \vDash (x, a) \circledast \ \vDash (x, b),$$

$$\forall x \in X,\ \ \vDash (x, \top) = \top.$$

ES3: P and $\vDash$ together satisfy *compatibility/enrichment axiom*:

$$\forall x, y \in X,\ \forall a \in A,\ \ P(x, y) \otimes \ \vDash (y, a) \ \leq \ \vDash (x, a).$$

Comment. This definition formulates programming question given above.

Examples. Later.

Enriched extents of predicates. $(L, \leq, \otimes, \circledast, e)$ unital complete po-ringoid, $(A, \leq, \otimes)$ complete po-groupoid, $(X, P, A, \vDash)$ L-enriched topological system. Put $ext_L : (A, \leq, \otimes) \to (L, \leq, \circledast)^X$

$$ext_L(a) : X \to L \quad \text{by} \quad ext_L(a)(x) = \ \vDash(x, a)$$

Then ext_L preserves arbitrary $\bigvee$, preserves $\otimes$ of A to pointwise lifted $\circledast$ of L, preserves $\top_A$ to $\underline{\top_L}$. **<u>BUT</u>** ES3 above implies new property for ext_L:

$$\forall a \in A,\ \forall x, y \in X,\ P(x, y) \otimes ext_L(a)(y) \leq ext_L(a)(x)$$

Have *extent space* $(X, (ext_L)^{\to}(A))$, where $(ext_L)^{\to}(A) \subset L^X$ satisfies: closed under arbitrary $\bigvee$, closed under $\circledast$ (lifted pointwise), contains $\underline{\top_L}$, and satisfies this <u>compatibility/enrichment condition</u>:

$$\forall\, u \in (ext_L)^{\to}(A),\ \forall x, y \in X,\ P(x, y) \otimes u(y) \leq u(x)$$

Enriched Topology (spaces). Let $(L, \leq, \otimes, \circledast, e)$ be unital complete po-ringoid. Then (X, P, τ) is *L-enriched/preordered topological space* if (X, P) is $(L, \leq, \otimes)$-preordered set and $\tau \subset L^X$ such that τ is:

SP1: closed under arbitrary $\bigvee$
SP2: closed under $\circledast$
SP3: contains $\underline{\top_L}$
SP4: $\forall\, u \in \tau,\ \forall x, y \in X,\ P(x, y) \otimes u(y) \leq u(x)$ (compatibility/enrichment)

Enriched Topology (fixed-basis). Let $(L, \leq, \otimes, \circledast, e)$ be unital complete left-residuated po-ringoid. Then *L*-**EnrTop** comprises all spaces (X, P, τ) from above together with all morphisms of the form $f : (X, P, \tau) \to (Y, Q, \sigma)$ such that $f : (X, P) \to (Y, Q)$ in *L*-**PreSet** and $f : (X, \tau) \to (Y, \sigma)$ in *L*-**Top**.

Theorem. Let $(L, \leq, \otimes, \circledast, e)$ be unital complete left-residuated po-ringoid. Then *L*-**EnrTop** is topological over each of *L*-**PreSet** and **Set** w.r.t. the expected forgetful functors.

Examples. Later.

Crisp/Many-Valued Specialization Preorders of Many-Valued Spaces

Recall. For $(X, \mathfrak{T})$ be topological space: $x \le_{\mathfrak{T}} y \Leftrightarrow y \in \overline{\{x\}}$. Equivalent to say:

$$x \le_{\mathfrak{T}} y \Leftrightarrow \forall U \in \mathfrak{T}, y \in U \Rightarrow x \in U$$

Defn: *L*-Specialization Order. Let L be complete po-groupoid and (X, τ) be L-topological space. Put: $x \le_\tau y \Leftrightarrow \forall u \in \tau,\ u(y) \le u(x)$. Also dual order.

Defn: *L*-Valued Specialization Orders. Let L be right-residuated complete po-monoid and (X, τ) be L-topological space. Put $P_\tau : X \times X \to L$ by

$$P_\tau(x, y) = \bigwedge_{u \in \tau} (u(x) \swarrow u(y))$$

If L left-residuated complete po-monoid, then put "dual" $Q_\tau : X \times X \to L$ by

$$Q_\tau(x, y) = \bigwedge_{u \in \tau} (u(x) \searrow u(y))$$

Note. $\le_\tau$ and dual are (crisp) preorders; P_τ and Q_τ are L-preorders.

Theorem. Crisp orders induced by L-valued specialization orders are precisely the crisp L-specialization orders.

Examples (spaces to enriched spaces to enriched systems). Let $(L, \leq, \otimes, \circledast, e)$ be a complete right-residuated po-ringoid such that $(L, \leq, \otimes)$ is a monoid; let (X, τ) be $(L, \leq, \circledast)$-topological space.

(1) Consider L-valued specialization order $P_\tau : X \times X \to (L, \leq, \otimes)$. Then P_τ is compatible with (X, τ), i.e., (X, P_τ, τ) is enriched L-topological space.

(2) Continuing from (1), put $\vDash_\tau : X \times \tau \to L$ by

$$\vDash_\tau (x, u) = u(x)$$

Then $(X, P_\tau, \tau, \vDash_\tau)$ is enriched $(L, \leq, \otimes)$-topological system.

(3) "Dualize" (1,2) using Q_τ if L left-residuated equipped with right partial distrib. law, order of pretensorands in compatibility axioms reversed.

Examples. Let $(L, \leq, \otimes, \circledast, e)$ be unital complete po-ringoid, $(A, \leq, \otimes)$ be complete po-groupoid, $(X, P, A, \vDash)$ be enriched L-topological system. Then $(X, P, ext_L^{\rightarrow}(A))$ is enriched L-topological space using satisfaction relation from (2) above.

Examples (systems to enriched systems). Let $(L,\leq,\otimes,\circledast,e)$ be complete right-residuated po-ringoid such that $(L,\leq,\otimes)$ monoid, $(A,\leq,\otimes)$ complete po-groupoid; let $(X,A,\vDash)$ be L-topological system. Then $\left(X,P_{ext_L^{\rightarrow}(A)},A,\vDash\right)$ is enriched L-topological system. "Dualize" using $Q_{ext_L^{\rightarrow}(A)}$ and related concepts and satisfacton relation from (2) above.

Examples (spectra to enriched systems/spaces). Let $(L,\leq,\otimes,\circledast,e)$ be complete right-residuated po-ringoid such that $(L,\leq,\otimes)$ monoid, $(A,\leq,\otimes)$ complete po-groupoid. Put:

$$Lpt(A)=\{p:A\to L \mid p \text{ preserves } \bigvee,\ \otimes \text{ to } \circledast,\ \top\},$$

$$P_A : Lpt(A)\times Lpt(A)\to L \text{ by } P_A(p,q)=\bigwedge_{a\in A}(p(a)\swarrow q(a)),$$

$$\vDash_A\ : Lpt(A)\times A\to L \text{ by } \vDash_A(p,a)=p(a)$$

Then $(Lpt(A),P_A,A,\vDash_A)$ is enriched L-topological system. For corresponding enriched L-topological extent space $(Lpt(A),P_A,ext_L^{\rightarrow}(A))$, $P_A=P_{ext_L^{\rightarrow}(A)}$. "Dualize" using $Q_A, Q_{ext_L^{\rightarrow}(A)}$ and related concepts.

Examples ((bit)strings from alphabets). Let:

Σ be set with $|\Sigma| \geq 2$, viewed as "alphabet"

$\Sigma^{*\omega} =: \{$countable strings of letters from $\Sigma\}$

$\mathbf{B} =: \mathbf{2}^{\omega} = \{$countably infinite strings of letters from $\mathbf{2}\}$ —complete Bool. alg. with $\otimes = \circledast = \wedge$

Put $P : \Sigma^{*\omega} \times \Sigma^{*\omega} \to \mathbf{B}$ by—for $n \in \mathbb{N}$—

$$P(\sigma_1, \sigma_2)(n) = \begin{cases} 1, & \sigma_1(n), \sigma_2(n) \text{ exist and } \sigma_1(n) = \sigma_2(n) \\ 0, & \text{otherwise} \end{cases}$$

Each $P(\sigma_1, \sigma_2)$ is *comparison bitstring*. It follows that $(\Sigma^{*\omega}, P)$ is $\mathbf{B}$-preordered set.

Now for $a \in \Sigma$, put $p^a : \Sigma^{*\omega} \to \mathbf{B}$ by

$$p^a(\sigma)(n) = \begin{cases} 1, & \sigma(n) \text{ exists and } \sigma(n) = a \\ 0, & \text{otherwise} \end{cases}$$

Each $p^a(\sigma)$ is *indicator string* and member of $\mathbf{B}^{\Sigma^{*\omega}}$. Now put

$$\mathbf{Q} = \langle\langle\{p^a : a \in \Sigma\}\rangle\rangle \subset \mathbf{B}^{\Sigma^{*\omega}},$$

the **B**-topology having subbasis the indicator strings. It can be shown that **Q** is compatible with P; hence $(\Sigma^{*\omega}, P, \mathbf{Q})$ is **B**-enriched topological space.

Finally, $(\Sigma^{*\omega}, P, \mathbf{Q}, \vDash_{\mathbf{Q}})$ is **B**-enriched topological system, where $\vDash_{\mathbf{Q}} : X \times \mathbf{Q} \to L$ by

$$\vDash_{\mathbf{Q}} (x, u) = u(x)$$

"String" spaces encountered again below w.r.t. many-valued T_1 separation.

Antisymmetry & L–T_0 Many-Valued Topological Spaces

Defn (antisymmetry and partial orders). Let L be a unital po-groupoid. Then $P : X \times X \to L$ on X is (L-)*antisymmetric* if

$$\forall x, y \in X, \;\; P(x,y) \geq e, \; P(y,x) \geq e \;\Rightarrow\; x = y;$$

and L-*partial order* is antisymmetric L-preorder; and L-**Poset** is full subcategory of L-**PreSet** comprising all L-posets.

Comment. Above definition justified in several directions: skeleton of each L-**PreSet**; quotients of L-presets (construction requires L-antisymmetry, IIA operators/involutions, and L-symmetry—latter below with L-T_1 issues); furnishes right-adjoint of each L-**PreSet**; generalizes classical result that **Poset** is monotopological construct; characterizes fundamental L-T_0 axiom in many-valued topology.

Theorem. For each complete unital po-groupoid L, L-**Poset** is monotoplogical over **Set** w.r.t. expected forgetful functor; so **Poset** is monotopological construct.

Defn (T_0 separation). Let L be complete po-groupoid and (X,τ) be L-topological space. Then (X,τ) is L-T_0 if

$$\forall x, y \in X,\ x \neq y \Rightarrow \exists u \in \tau,\ u(x) \neq u(y)$$

The L-T_0 axiom has different formulations (e.g., injectivity of the L-Stone second comparison maps), and is well-established via representation theorems and compactification reflectors, including two successful forms of sobriety. Examples include spectra and the fuzzy real lines and unit intervals. More justification below.

Main Theorem. Assume: L complete po-groupoid, L-topological space (X,τ).

(1) $\leq_\tau$ (and dual) is antisymmetric (and po) iff (X,τ) is L-T_0.

(2) Further assume L right-residuated complete po-monoid. Then $P_\tau : X \times X \to L$ is antisymmetric (and po) iff (X,τ) is L-T_0.

(3) Further assume L left-residuated complete po-monoid. Then $Q_\tau : X \times X \to L$ is antisymmetric (and po) iff (X,τ) is L-T_0.

(4) For L a right[left]-residuated complete po-monoid, $\leq_\tau$ (and dual) is antisymmetric iff P_τ [Q_τ] is antisymmetric. Hence for L unital quantale, $\leq_\tau$ (and dual) is antisymmetric iff P_τ is antisymmetric iff Q_τ is antisymmetric.

(5) For L DeMorgan frame with antitone involution $' : L \to L$, the L-valued hemimetric $P'_\tau : X \times X \to L$ induced by P_τ satisfies the following positive definiteness condition—

$$[\forall x, y \in X,\ P'_\tau(x,y) = P'_\tau(y,x) = \bot \Leftrightarrow x = y]$$

if and only if (X,τ) is L-T_0.

Notes.

(1) *L*-antisymmetry essentially "same as" traditional antisymmetry, so appropriate generalization.

(2) To check *L*-antisymmetry of P_τ, suffices to check antisymmetry of $\leq_\tau$. But latter always included in former (for right-residuated complete po-monoid case) since $\chi^e_{\leq_\tau} \leq P_\tau$, where $\chi^e_{\leq_\tau} \equiv \chi_{\leq_\tau} \wedge \underline{e} : X \times X \to \{\perp, e\} \subset L$.

Symmetry & L–T_1(1) / L–T_1(2) Separation in Many-Valued Topology

Recall. For L unital groupoid, P is L-valued preorder on X if:

P0: $P : X \times X \to L$ is mapping (degrees of precedence)
P1: $\forall x \in X,\ e \le P(x,x)$ (reflexivity)
P2: $\forall x, y, z \in X,\ P(x,y) \otimes P(y,z) \le P(x,z)$ (transitivity)

Can also consider:

P3: $\forall x, y \in X,\ P(x,y) \ge e,\ P(y,x) \ge e \Rightarrow x = y$ (antisymmetry)

Defn (symmetry). Let X be set and $(L, \le, \otimes, e, {}^*)$ be unital IIA po-groupoid. Then P is a symmetric L-valued relation on X if P satisfies P0 and

P4: $\forall x, y \in X,\ P(x,y) = P^*(y,x)$ (symmetry),

${}^* : L \to L$ is IIA oper./invol. ($a^{**} = a,\ a \le b \Rightarrow a^* \le b^*,\ (a \otimes b)^* = b^* \otimes a^*$).

Remark. If $\otimes$ is commutative and * is chosen as id_L, then P4 becomes

$$\forall x, y \in X,\ P(x,y) = P(y,x)$$

Enriched Topology and Asymmetry

Note. Many-valued symmetry above central to many-valued antisymmetry capturing quotients of L-presets as L-posets and to the latter comprising the right adjoint of the former. Important justification of many-valued symmetry.

Strategy. Know antisymmetry characterizes T_0 for traditional topology, symmetry characterizes T_1 for traditional topology, and antisymmetry (crisp or many-valued) characterizes L-T_0 for many-valued topology. Propose to *define* L-T_1 for many-valued topology using above many-valued symmetry. Accordingly, propose to define *asymmetry* for many-valued topology as the denial of above many-valued symmetry.

Defn. For L a right-residuated complete po-monoid, an L-topological space (X, τ) or its topology τ is *asymmetric* if $\leq_\tau$ or P_τ is not symmetric.

Typically, asymmetric space is also L-T_0.

Standing Assumption. Until stated otherwise, L in sequel is unital IIA quantale.

IIA induced spaces. Let (X, τ) be L-topological space. Put:

$$\tau^* = \{u^* : u \in \tau\}, \quad T = \tau \vee \tau^*$$

Then $(X, \tau^*), (X, T)$ are L-topological spaces; and T is smallest *-invariant topology containing τ (so $T^* = T$).

Example. Suppose $\otimes$ non-commutative with $^* \neq id_L$; $\exists a \in L$, $a^* \neq a$. Put

$$\tau = \{\underline{\bot}, \underline{a}, \underline{\top}\}$$

for some set X. Then (X, τ) is L-topological space and $\tau \subsetneqq T$.

Lemma. Let X be set and $x, y \in X$.

(1) $P_\tau^* : X \times X \to L$ by $P_\tau^*(x, y) = \bigwedge_{u \in \tau} (u^*(y) \searrow u^*(x))$.

(2) $P_\tau^*(x, y) = Q_{\tau^*}(y, x)$.

(3) P_τ symmetric iff $P_\tau = Q_{\tau^*}$; Q_τ symmetric iff $Q_\tau = P_{\tau^*}$; P_τ symmetric iff Q_τ symmetric.

(4) P_T symmetric iff $P_T = Q_T$ iff Q_T symmetric.

Theorem (L-specialization *vis-a-vis* L-valued specialization w.r.t. symmetry).

(1) Let P_τ be symmetric. Then:

  (a) $\leq_\tau$ symmetric;

  (b) $\leq_{\tau^*}$ coincides with $\leq_\tau$ and hence symmetric;

  (c) $\leq_T$ coincides with $\leq_\tau$ and hence symmetric.

(2) Converse to (1)(a) fails, even when $\leq_\tau$ additionally assumed antisymmetric, i.e., (X,τ) also L-T_0.

Examples.

(1) Let $X=\{x,y\}, L=\{\bot,a,b,\top\}$: $\otimes=\wedge$, $\searrow\ =\ \swarrow\ =\ \rightarrow$, $^*=id_L$, so $P_\tau^*(y,x)$ $=P_\tau(y,x)$. Put: $u(x)=\bot, u(y)=a$; $v(x)=b, v(y)=\top$; $o(x)=b, o(y)=a$; $\tau=\{\underline{\bot},\underline{u},\underline{v},\underline{o},\underline{\top}\}$. Then (X,τ) is L-topological space which is L-T_0 and for which $\leq_\tau$ is both antisymmetric and symmetric (and hence trivial), and P_τ is L-antisymmetric but <u>not</u> L-symmetric.

(2) As in (1), but not include $\underline{o}$. Then (X,τ) is L-topological space which is L-T_0 and for which $\leq_\tau$ is antisymmetric but <u>not</u> symmetric, and P_τ is L-antisymmetric but <u>not</u> L-symmetric.

Defn (L-T_1 separation axioms). Let (X,τ) be L-topological space.

(1) Suppose L semiquantale. (X,τ) is L-T_1 in *first sense*, or L-$T_1(1)$, if $\leq_\tau$ antisymmetric and symmetric.

(2) Suppose L unital IIA quantale. (X,τ) is L-T_1 in *second sense*, or L-$T_1(2)$, if P_τ antisymmetric and symmetric. So, (X,τ) is L-$T_1(2)$ if P_τ satisfies:

P1: $\forall x \in X,\ P_\tau(x,x) \geq e$;

P2: $\forall x,y,z \in X,\ P_\tau(x,y) \otimes P_\tau(y,z) \leq P_\tau(x,z)$;

P3: $\forall x,y \in X,\ P_\tau(x,y) \geq e,\ P_\tau(y,x) \geq e \ \Rightarrow\ x = y$;

P4: $\forall x,y \in X,\ P(x,y) = P^*(y,x)$.

Corollary. Let L be unital IIA quantale and (X,τ) be L-topological space.

(1) L-$T_1(1)$ implies L-T_0, but not conversely.

(2) L-$T_1(2)$ implies L-T_0, but not conversely.

(3) L-$T_1(2)$ implies L-$T_1(1)$, but not conversely.

Comment. When evaluating an L-topological space for asymmetry, it merely suffices to determine if $\leq_T$ is not symmetric; and if the space is L-T_0, then it suffices to determine that the space is not L-$T_1(1)$.

Reconciliation with Kubiak (1995) L-T_1 **axiom**. Let L be semiquantale and (X,τ) be L-topological space.

(1) (X,τ) is L-$T_1(K)$ if:

$$\forall x,y \in X,\ x \neq y \Rightarrow \exists\, u,v \in \tau,\ u(y) \not\leq u(x) \text{ and } v(x) \not\leq v(y)$$

(2) Rewrite L-$T_1(K)$:

$$\forall x,y \in X,\ x \neq y \Rightarrow x \not\leq_\tau y \text{ and } y \not\leq_\tau x. \quad \text{(K1)}$$

(3) Rewrite L-$T_1(1)$:

$$\forall x,y \in X,\ x \not\leq_\tau y \Leftrightarrow y \not\leq_\tau x. \quad \text{(K2)}$$

(4) Assume L-$T_1(K)$, suppose $x \leq_\tau y$. Then (K1) gives $x = y$, reflexivity of $\leq_\tau$ gives $y \leq_\tau x$. So L-$T_1(1)$ holds.

(5) Assume L-$T_1(1)$, suppose $x \neq y$. Antisymmetry of $\leq_\tau$ yields $x \not\leq_\tau y$ or $y \not\leq_\tau x$. But (K2) in each case yields $x \not\leq_\tau y$ and $y \not\leq_\tau x$. By (K1), L-$T_1(K)$ holds.

(6) So L-$T_1(1) \Leftrightarrow L$-$T_1(K)$. Different motivations: L-$T_1(K)$ motivated by giving symmetric version of L-T_0; L-$T_1(1)$ motivated by symmetry of L-specialization order $\leq_\tau$.

Spectra Related Examples. Let **SQuant** be category of all semiquantales and all morphisms being those maps preserving arbitrary $\bigvee$ and $\otimes$; and let **Squant**$_\top$ be subcategory of all semiquantales and those semiquantale morphisms which also present $\top$.

Fix semiquantale *L*, and let *A* be any semiquantale. Put:

$$Lpt(A) = \mathbf{SQuant}_\top(A, L) = \{p : A \to L \mid p \text{ preserves } \bigvee, \otimes, \top\},$$

$$\Phi_L : A \to L^{Lpt(A)} \text{ by } \Phi_L(a) : Lpt(A) \to L \text{ by } \Phi_L(a)(p) = p(a).$$

Then *LPt*(*A*) = $(Lpt(A), (\Phi_L)^{\to}(A))$ is *L*-topological space, the *L-spectrum* of *A*.

Defn. Let *A* be semiquantale.

(1) $c \in A - \{\top\}$ is $\otimes$-*prime* if: $\forall a, b \in A,\ a \otimes b \le c \Leftrightarrow a \le c$ or $b \le c$. $Pr_\otimes(A)$ set of all $\otimes$-primes of *A*.

(2) *A* has *two related* $(\otimes$-$)$*primes* if $\exists\, a, b \in Pr_\otimes(A),\ a \le b$ and $a \ne b$.

Lemma. *LPt*(*A*) is *L*-$T_1(1)$ iff $\forall p, q \in Lpt(A),\ q \le p \Leftrightarrow p \le q$ (in *Lpt*(*A*)).

Theorem. Assume *L* integral, $\perp$ annihilator for $\otimes$ in L, A. Then *L* consistent & *A* has two related primes $\Rightarrow$ *LPt*(*A*) not *L*-$T_1(1)$. "Iff" if $L = \mathbf{2}$.

Example. Assume L integral & consistent, $\perp$ annihilator for $\otimes$ in L, A, and $\mathfrak{T}_{cof}$ cofinite topology of $\mathbb{R}$. Then $LPt(\mathfrak{T}_{cof})$ not L-$T_1(1)$.

Examples. Let L be consistent, complete DeMorgan algebra; let $\mathbb{R}(L), \mathbb{R}_l(L), \mathbb{R}_r(L), \mathbb{I}(L), \mathbb{I}_l(L), \mathbb{I}_r(L)$ be L-fuzzy real line, L-fuzzy left-handed real line, L-fuzzy right-handed real line, L-fuzzy unit interval, L-fuzzy left-handed unit interval, L-fuzzy right-handed unit interval.

(1) All are L-T_0.

(2) $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are L-$T_1(1)$; the others are not L-$T_1(1)$.

(3) Recall $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$, so-called "alternative" fuzzy real line and unit interval, are defined to be $LPt(\mathfrak{T})$, where $\mathfrak{T}$ is standard topology on $\mathbb{R}$ and $\mathbb{I}$, respectively. For L a complete Boolean algebra, $\mathbb{R}^*(L)$ and $\mathbb{I}^*(L)$ are both L-sober and L-$T_1(1)$.

(4) For L a complete Boolean algebra, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are both L-sober and L-$T_1(1)$. (This follows from (3)).

Examples. Recall **B**-enriched topological spaces $(\Sigma^{*\omega}, P, \mathbf{Q})$ described above.

(1) P is reflexive, transitive, compatible, antisymmetric, symmetric.

(2) **B**-valued specialization order $P_{\mathbf{Q}} = P$. Hence $(\Sigma^{*\omega}, P, \mathbf{Q})$ is L-$T_1(2)$.

Examples (behavior w.r.t. $M_L \dashv G_\chi$). For L-T_0 and L-$T_1(1)$ issues, L is semiquantale; for L-$T_1(2)$ issues, L is integral IIA quantale.

(1) $(X,\mathfrak{T})$ T_0 iff $G_\chi(X,\mathfrak{T})$ L-T_0; $(X,\mathfrak{T})$ T_1 iff $G_\chi(X,\mathfrak{T})$ L-$T_1(1)$.

(2) M_L reflects T_0 [T_1] to L-T_0 [L-$T_1(1)$].

(3) G_χ preserves T_1 to L-$T_1(2)$, and reflects L-$T_1(2)$ to T_1. Hence G_χ preserves asymmetry.

(4) M_L reflects T_1 to L-$T_1(2)$, and hence preserves L-asymmetry.

Examples (behavior w.r.t. $\omega_L \dashv \iota_L$). Generally, L is semiquantale.

(1) Let $(X,\tau) \in |L\text{-}\mathbf{Top}|$. Then (X,τ) is L-$T_1(1)$ iff $\iota_L(X,\tau)$ is T_1. Hence ι_L reflects and preserves L-asymmetry.

(2) Let $(X,\mathfrak{T}) \in |\mathbf{Top}|$. Then $(X,\mathfrak{T})$ is T_1 implies $\omega_L(X,\mathfrak{T})$ is L-$T_1(1)$, in which case ω_L reflects L-asymmetry; and the converse holds if L is completely distributive ($\otimes = \wedge$), in which case ω_L preserves asymmetry.

(3) Continuing (2), if L admits an endomorphism other than the identity, then for $(X,\mathfrak{T})$ being T_1, $\omega_L(X,\mathfrak{T})$ is both L-$T_1(1)$ and <u>not</u> L-sober.

Other Separation Issues.

(1) Strong L-T_2 (Höhle) implies weak L-T_2 (Kubiak) implies L-$T_1(1)$. But neither Hausdorff axiom implies L-sober: for any L complete DeMorgan frame which is non Boolean, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are strong L-T_2 but not L-sober.

(2) For L semiquantale, have: ι_L-Hausdorff implies L-$T_1(1)$ implies L-T_0, and ι_L-Hausdorff implies ι_L-sober implies L-T_0, and L-$T_1(1)$ and ι_L-sober unrelated. For L completely distributive DeMorgan algebra, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are ι_L-Hausdorff and ι_L-sober and L-$T_1(1)$; but if L is also non-Boolean, $\mathbb{R}(L)$ and $\mathbb{I}(L)$ are not L-sober. So ι_L-Hausdorff $\nRightarrow$ L-sobriety.

Example (non-commutative conjunctions in programming / web searches).

Two numerical variables *x*, *y* related to website.

(1) *y* counts how many times website has been visited.
(2) Whenever *y* is used in expression, website is accessed before value in *y* is read, so *y* always updated.
(3) Each time *y* is read, its value increases by 1.
(4) Predicates related to website take on traditional truth values **T** or **F**. The conjunction of predicates φ, ψ written $\varphi \sqcap \psi$, which is **T** if and only if each of φ, ψ is **T**.
(5) Website not accessed when conjunctions of associated predicates are formed.
(6) Conjunction $\varphi \sqcap \psi$ read left-to-right, first φ, then ψ.
(7) Assume:

current value of *x* is 9, current value of *y* is 8

Consider predicates $P : [x = y],\ Q : [y \geq 10]$.

(8) Truth value of $P \sqcap Q$ is **T**: reading P first updates y to $y = 9$; reading Q second updates y yet again, giving $y = 10$; so each of P, Q is **T**.

(9) Truth value of $Q \sqcap P$ is **F**: reading Q first updates y to $y = 9$; reading P second updates y yet again, giving $y = 10$; so each of P, Q is **F**.

(10) Note also that $P \sqcup Q$ is **T**, while $Q \sqcup P$ is **F**.

(11) Note if Q is replaced by $\hat{Q} : [y \geq 9]$, then:

$$P \sqcap \hat{Q} \text{ is } \mathbf{T}, \quad \hat{Q} \sqcap P \text{ is } \mathbf{F}$$

$$P \sqcup \hat{Q} \text{ is } \mathbf{T}, \quad \hat{Q} \sqcup P \text{ is } \mathbf{T}$$

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