6-2017

On Cohomological Dimensions of Remainders of Stone-Čech Compactifications

Vladimer Baladze
Batumi Shota Rustaveli State University, vbaladze@gmail.com

On Cohomological Dimensions of Remainders of Stone-Čech Compactifications

Vladimer Baladze

Department of Mathematics
Batumi Shota Rustaveli State University

Abstract

In the paper the Čech border homology and cohomology groups of closed pairs of normal spaces are constructed and investigated. These groups give intrinsic characterizations of Čech homology and cohomology groups based on finite open coverings, small and large cohomological dimensions of remainders of Stone-Čech compactifications of metrizable spaces.

Keywords and Phrases: Čech homology, Čech cohomology, Stone-Čech compactification, remainder, cohomological dimension.

Introduction

The investigation and discussion presented in this paper are centered around the following problem:

Find necessary and sufficient conditions under which a space of given class has a compactification whose remainder has the given topological property (cf. [Sm_2], Problem I, p.332 and Problem II, p.334).

This problem for different topological invariants and properties was studied by several authors:

- J.M.Aarts [A], J.M.Aarts and T.Nishiura [A-N], Y. Akaike, N. Chinen and K. Tomoyasu [Ak-Chin-T], V.Baladze [B_1], M.G. Charalambous [Ch], A.Chigogidze ([Chi_1], [Chi_2]), H. Freudenthal ([F_1],[F_2]), K.Morita [Mo], E.G. Skljarenko [Sk], Ju.M.Smirnov ([Sm_1]-[Sm_5]) and H.De Vries [V] found conditions under which the spaces have extensions whose remainders have given covering and inductive dimensions, and combinatorial properties.

- The remainders of finite order extensions are defined and investigated by H.Inasaridze ([I_1], [I_2]). Using the results obtained in these papers, H.Inasaridze [I_3], L.Zambakhidze ([Z_1],[Z_2]), and I.Tsereteli [Ts] solved interesting problems of homological algebra, general topology and dimension theory.

- n-dimensional (co)homology groups of remainders of precompact spaces are studied by V.Baladze [B_3], V.Baladze and L.Turmanidze [B-Tu].
- A.Calder [C] described n-dimensional cohomotopy groups of remainders of Stone-Čech compactifications.
- The characterizations of shapes of remainders of spaces are established in papers of V.Baladze ([B_2],[B_3]), B.J.Ball [Ba], J.Keesling ([K_1], [K_2]), J.Keesling and R.B. Sher [K-Sh].

The present paper is motivated by the general problem mentioned above. Specifically, we study this problem for the properties: Čech (co)homology groups based on finite open covers and cohomological dimensions of remainders of Stone-Čech compactifications of metrizable spaces are given groups and given numbers, respectively.

In this paper we define the Čech type covariant and contravariant functors which coefficients in an abelian group G,

$$\check{\mathrm{H}}_n^{\infty}(-,-;G) : \mathscr{N}_p^{\mathbf{2}} \to \mathscr{A}b$$

and

$$\hat{\mathrm{H}}^n_{\infty}(-,-;G) : \mathscr{N}_p^{\mathbf{2}} \to \mathscr{A}b,$$

from the category $\mathscr{N}_p^2$ of closed pairs of normal spaces and proper maps to the category $\mathscr{A}b$ of abelian groups and homomorphisms. The construction of these functors is based on all border open covers of pairs $(X, A) \in ob(\mathscr{N}_p^{\mathbf{2}})$ (see Definition 1.1 and Definition 1.2).

One of our main results of the paper is the following theorem (see Theorem 2.1). Let $\mathscr{M}_p^2$ be the category of closed pairs of metrizable spaces and proper maps. For each closed pair $(X, A) \in ob(\mathscr{M}_p^{\mathbf{2}})$, one has

$$\check{H}_n^f(\beta X \setminus X, \beta A \setminus A; G) = \check{H}_n^{\infty}(X, A; G)$$

and

$$\hat{H}^n_f(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}^n_{\infty}(X, A; G),$$

where $\check{H}_n^f(\beta X \setminus X, \beta A \setminus A; G)$ and $\hat{H}^n_f(\beta X \setminus X, \beta A \setminus A; G)$ are Čech homology and cohomology groups based on all finite open covers of $(\beta X \setminus X, \beta A \setminus A)$, respectively (see [E-St], Ch. IX, p.237).

We also consider the border small and large cohomological dimensions $d^f_{\infty}(X; G)$ and $D^f_{\infty}(X; G)$ and prove the following relations (see Theorem 2.5 and Theorem 2.8):

$$d^f_{\infty}(X; G) = d_f(\beta X \setminus X; G),$$

$$D^f_{\infty}(X; G) = D_f(\beta X \setminus X; G),$$

where $d_f(\beta X \setminus X; G)$ and $D_f(\beta X \setminus X; G)$ are small cohomological dimension and large cohomological dimension [N] of remainder $(\beta X \setminus X, \beta A \setminus A)$, respectively.

1 On Čech border homology and cohomology groups

In this section we give an outline of a generalization of Čech homology theory by replacing the set of all finite open coverings in the definition of Čech (co)homology group $(\hat{H}^n_f(X,A;G))$ $\check{H}^f_n(X,A;G)$ (see [E-St],Ch.IX, p.237) by a set of all finite open families with compact enclosures. For this aim we give the following definitions.

An indexed family of subsets of set X is a function α from an indexed set V_α to the set 2^X of subsets of X. The image $\alpha(v)$ of index $v \in V_\alpha$ is denoted by α_v. Thus the indexed family α is the family $\alpha = \{\alpha_v\}_{v\in V_\alpha}$. If $|V_\alpha| < \aleph_0$, then we say that α family is a finite family.

Let $V_\alpha^{'}$ be a subset of set V_α. A family $\{\alpha_v\}_{v\in V'_\alpha}$ is called a subfamily of family $\{\alpha_v\}_{v\in V_\alpha}$.

By $\alpha = \{\alpha_v\}_{v\in (V_\alpha, V'_\alpha)}$ we denote the family consisting of family $\{\alpha_v\}_{v\in V_\alpha}$ and its subfamily $\{\alpha_v\}_{v\in V'_\alpha}$.

Definition 1.1. (see [Sm$_4$]). A finite family $\alpha = \{\alpha_v\}_{v\in V_\alpha}$ of open subsets of normal space X is called a border cover of X if its enclosure $K_\alpha = X \setminus \bigcup_{v\in V_\alpha} \alpha_v$ is a compact subset of X.

Definition 1.2. (cf. [Sm$_4$]). A finite open family $\alpha = \{\alpha_v\}_{v\in (V_\alpha, V^A_\alpha)}$ is called a border cover of closed pair $(X,A) \in \mathscr{N}^2$ if there exists a compact subset K_α of X such that $X \setminus K_\alpha = \bigcup_{v\in V_\alpha} \alpha_v$ and $A \setminus K_\alpha \subseteq \bigcup_{v\in V^A_\alpha} \alpha_v$.

The set of all border covers of (X,A) is denoted by $\text{cov}_\infty(X,A)$. Let $K^A_\alpha = K_\alpha \cap A$. Then the family $\{\alpha_v \cap A\}_{v\in V^A_\alpha}$ is a border cover of subspace A.

Definition 1.3. Let $\alpha, \beta \in \text{cov}_\infty(X,A)$ be two border covers of (X,A) with indexing pairs (V_α, V^A_α) and (V_β, V^A_β), respectively. We say that the border cover β is a refinement of border cover α if there exists a refinement projection function $p : (V_\beta, V^A_\beta) \to (V_\alpha, V^A_\alpha)$ such that for each index $v \in V_\beta$ ($v \in V^A_\beta$) $\beta_v \subset \alpha_{p(v)}$.

It is clear that $\text{cov}_\infty(X,A)$ becomes a directed set with the relation $\alpha \leq \beta$ whenever β is a refinement of α.

Note that for each $\alpha \in \text{cov}_\infty(X,A)$, $\alpha \leq \alpha$, and if for each $\alpha,\beta,\gamma \in \text{cov}_\infty(X,A)$, $\alpha \leq \beta$ and $\beta \leq \gamma$, then $\alpha \leq \gamma$.

Let $\alpha, \beta \in \text{cov}_\infty(X,A)$ be two border covers with indexing pairs (V_α, V^A_α) and (V_β, V^A_β), respectively. Consider a family $\gamma = \{\gamma_v\}_{v\in(V_\gamma, V^A_\gamma)}$, where $V_\gamma = V_\alpha \times V_\beta$ and $V^A_\gamma = V^A_\alpha \times V^A_\beta$. Let $v = (v_1, v_2)$, where $v_1 \in V_\alpha$, $v_2 \in V_\beta$. Assume that $\gamma_v = \alpha_{v_1} \cap \beta_{v_2}$. The family $\gamma = \{\gamma_v\}_{v\in(V_\gamma, V^A_\gamma)}$ is a border cover of (X,A) and $\gamma \geq \alpha, \beta$.

For each border cover $\alpha \in \text{cov}^\infty(\text{X},\text{A})$ with indexing pair (V_α, V^A_α), by (X_α, A_α) denote the nerve α, where A_α is the subcomplex of simplexes s of complex X_α with vertices of V_α^A such that $\mathrm{Car}_\alpha(\mathrm{s}) \cap \mathrm{A} \neq \emptyset$, where $\mathrm{Car}_\alpha(s)$ is the carrier of simplex s (see [E-St], pp.234). The pair (X_α, A_α) is a simplicial pair. Moreover, any two refinement projection functions $p, q : \beta \to \alpha$ induce contiguous simplicial maps of simplicial pairs $p_\alpha^\beta, q_\alpha^\beta : (X_\beta, A_\beta) \to (X_\alpha, A_\alpha)$ (see [E-St], pp. 234-235).

Using the construction of formal homology theory of simplicial complexes ([E-St], Ch.VI) we can define the unique homomorphisms

$$p_{\alpha *}^\beta : H_n(X_\beta, A_\beta : G) \to H_n(X_\alpha, A_\alpha; G)$$

and

$$p_\alpha^{\beta *} : H^n(X_\alpha, A_\alpha : G) \to H_n(X_\beta, A_\beta; G),$$

where G is any abelian coefficient group.

Note that $p_{\alpha *}^\alpha = 1_{H_n(X_\alpha, A_\alpha : G)}$ and $p_\alpha^{\alpha *} = 1_{H^n(X_\alpha, A_\alpha : G)}$. If $\gamma \geq \beta \geq \alpha$ than

$$p_{\alpha *}^\gamma = p_{\alpha *}^\beta \cdot p_{\beta *}^\gamma$$

and

$$p_\alpha^{\gamma *} = p_\beta^{\gamma *} \cdot p_\alpha^{\beta *}.$$

Thus, the families

$$\{H_n(X_\alpha, A_\alpha; G), p_{\alpha *}^\beta, \mathrm{cov}_\infty(X, A)\}$$

and

$$\{H^n(X_\alpha, A_\alpha; G), p_\alpha^{\beta *}, \mathrm{cov}_\infty(X, A)\}$$

form inverse and direct systems of groups.

The inverse and direct limit groups of above defined inverse and direct systems are denoted by symbols

$$\check{H}_n^\infty(X, A; G) = \varprojlim \{H_n(X_\alpha, A_\alpha; G), p_{\alpha *}^\beta, \mathrm{cov}_\infty(X, A)\}$$

and

$$\hat{H}_\infty^n(X, A; G) = \varinjlim \{H^n(X_\alpha, A_\alpha; G), p_\alpha^{\beta *}, \mathrm{cov}_\infty(X, A)\}$$

and called n-dimensional Čech border homology group and n-dimensional Čech border cohomology group of pair (X, A) with coefficients in abelian group G, respectively.

Now we define, for a given proper map $f : (X, A) \to (Y, B)$ of pairs, the induced homomorphisms

$$f_*^\infty : \check{H}_n^\infty(X, A; G) \to \check{H}_n^\infty(Y, B; G)$$

and

$$f_\infty^* : \hat{H}_\infty^n(X, A; G) \to \hat{H}_\infty^n(Y, B; G).$$

Let $\alpha \in \mathrm{cov}_\infty(Y, B)$ be a border cover with index set V_α and $K_\alpha = Y \setminus \bigcup_{v \in V_\alpha} \alpha_v$. Consider a family $\alpha' = \{f^{-1}(\alpha_v)\}_{v \in V_\alpha}$. Note that

$$X \setminus \bigcup_{v \in V_\alpha} f^{-1}(\alpha_v) = X \setminus f^{-1}(\bigcup_{v \in V_\alpha} \alpha_v) = X \setminus f^{-1}(Y \setminus K_\alpha) = f^{-1}(K_\alpha).$$

Let $\alpha_v^{'} = f^{-1}(\alpha_v)$ and $V_{\alpha'} = V_\alpha$. Since f is proper, $f^{-1}(K_\alpha)$ is a compact subset of X.

Since $B \setminus K_\alpha \subseteq \bigcup_{v \in V_\alpha^B} \alpha_v$, the subfamily $\{f^{-1}(\alpha_v) | v \in V_\alpha^B\}$ is such that $A \setminus f^{-1}(K_\alpha) \subseteq \bigcup_{v \in V_\alpha^B} f^{-1}(\alpha_v)$. Let $V_{\alpha'}^A = V_\alpha^B$ and $K_{\alpha'} = f^{-1}(K_\alpha)$. Note that $A \setminus K_{\alpha'} \subset \bigcup_{v \in V_{\alpha'}^A} f^{-1}(\alpha_v)$. Hence, $\alpha^{'} = \{f^{-1}(\alpha_v)\}_{v \in (V_{\alpha'}, V_{\alpha'}^A)}$ is a border cover of pair (X, A).

It is clear that $X_{\alpha'}$ is a subcomplex of Y_α and $A_{\alpha'}$ is a subcomplex of B_α. By a symbol $f_\alpha : (X_{\alpha'}, A_{\alpha'}) \to (Y_\alpha, B_\alpha)$ denote the simplicial inclusion of $(X_{\alpha'}, A_{\alpha'})$ into (Y_α, B_α).

If $\alpha, \beta \in \mathrm{cov}_\infty(Y, B)$ and $\beta \geq \alpha$, then the diagrams

$$\begin{array}{ccc}
H_n(X_{\beta'}, A_{\beta'}; G) & \xrightarrow{f_{\beta *}} & H_n(X_\beta, A_\beta; G) \\
\Big\downarrow p_{\alpha' *}^{\beta'} & & \Big\downarrow p_{\alpha *}^{\beta} \\
H_n(X_{\alpha'}, A_{\alpha'}; G) & \xrightarrow[f_{\alpha *}]{} & H_n(X_\alpha, A_\alpha; G)
\end{array}$$

and

$$\begin{array}{ccc}
H^n(X_\alpha, A_\alpha; G) & \xrightarrow{f_\alpha^*} & H^n(X_{\alpha'}, A_{\alpha'}; G) \\
\Big\downarrow p_\alpha^{\beta *} & & \Big\downarrow p_{\alpha'}^{\beta' *} \\
H^n(X_\beta, A_\beta; G) & \xrightarrow[f_\beta^*]{} & H^n(X_{\beta'}, A_{\beta'}; G).
\end{array}$$

commute.

Thus, for each $\alpha \in \mathrm{cov}_\infty(Y, B)$, the induced homomorphisms $f_{\alpha *}$ and f_α^* together with function $\varphi : \mathrm{cov}_\infty(Y, B) \to \mathrm{cov}_\infty(X, A)$ given by formula

$$\varphi(\alpha) = f^{-1}(\alpha), \alpha \in \mathrm{cov}_\infty(Y, B)$$

form maps

$$(f_{\alpha *}, \varphi) : \{H_n(X_{\alpha'}, A_{\alpha'}), p_{\alpha' *}^{\beta'}, \mathrm{cov}_\infty(X, A)\} \to \{H_n(Y_\alpha, A_\alpha), p_{\alpha *}^{\beta}, \mathrm{cov}_\infty(Y, B)\}$$

and

$$(f_\alpha^*, \varphi) : \{H^n(Y_\alpha, A_\alpha), p_\alpha^{\beta *}, \mathrm{cov}_\infty(Y,B)\} \to \{H^n(X_{\alpha'}, A_{\alpha'}), p_{\alpha'}^{\beta'_*}, \mathrm{cov}_\infty(X,A)\}.$$

The limits of maps $(f_{\alpha *}, \varphi)$ and (f_α^*, φ) are denoted by

$$f_*^\infty : \check{H}_n^\infty(X, A; G) \to \check{H}_n^\infty(Y, B; G)$$

and

$$f_\infty^* : \hat{H}_\infty^n(Y, B; G) \to \hat{H}_\infty^n(X, A; G)$$

and called homomorphisms induced by proper map $f : (X, A) \to (Y, B)$.

Note that if $f : (X, A) \to (Y, B)$ is the identity map, then the induced homomorphisms $f_*^\infty : \check{H}_n^\infty(X, A; G) \to \check{H}_n^\infty(Y, B; G)$ and $f_\infty^* : \hat{H}_\infty^n(Y, B; G) \to \hat{H}_\infty^n(X, A; G)$ are the identity homomorphisms. Furthermore, for each proper maps $f : (X, A) \to (Y, B)$ and $g : (Y, B) \to (Z, C)$

$$(g \cdot f)_*^\infty = g_*^\infty \cdot f_*^\infty$$

and

$$(g \cdot f)_\infty^* = f_\infty^* \cdot g_\infty^*.$$

We have the following theorem.

Theorem 1.4. *There exist the covariant and contravariant functors*

$$\check{\mathrm{H}}_*^\infty(-,-;G) : \mathscr{N}_p^2 \to \mathscr{A}b$$

and

$$\hat{\mathrm{H}}_\infty^*(-,-;G) : \mathscr{N}_p^2 \to \mathscr{A}b$$

given by formulas

$$\check{\mathrm{H}}_*^\infty(-,-;G)(X,A) = \check{H}_*^\infty(X, A; G), \quad (X, A) \in ob(\mathscr{N}_p^2)$$

$$\check{\mathrm{H}}_*^\infty(-,-;G)(f) = f_*^\infty, \quad f \in \mathrm{Mor}_{\mathscr{N}_p^2}((\mathrm{X},\mathrm{A}),(\mathrm{Y},\mathrm{B}))$$

and

$$\hat{\mathrm{H}}_\infty^*(-,-;G)(X,A) = \hat{H}_\infty^*(X, A; G), \quad (X, A) \in ob(\mathscr{N}_p^2)$$

$$\hat{\mathrm{H}}_\infty^*(-,-;G)(f) = f_\infty^*, \quad f \in \mathrm{Mor}_{\mathscr{N}_p^2}((\mathrm{X},\mathrm{A}),(\mathrm{Y},\mathrm{B})).$$

We will call the functors $\check{\mathrm{H}}_*^\infty(-,-;G)$ and $\hat{\mathrm{H}}_\infty^*(-,-;G)$ Čech border homology and cohomology functors, respectively.

Now we define boundary and coboundary homomorphisms

$$\partial_n^\infty : \check{H}_n^\infty(X, A; G) \to \check{H}_{n-1}^\infty(A; G)$$

and

$$\delta_\infty^n : \hat{H}_\infty^{n-1}(A; G) \to \hat{H}_\infty^n(X, A; G).$$

Let $(X, A) \in ob(\mathscr{N}_p^2)$, $\beta, \alpha \in \mathrm{cov}_\infty(X, A)$ and $\beta \geq \alpha$. The refinement projection functions induce the unique homomorphisms $p_{\alpha *}^\beta : H_n(A_\beta; G) \to H_n(A_\alpha; G)$ and $p_\alpha^{\beta *} : H^n(A_\alpha; G) \to H^n(A_\beta; G)$, $p_{\alpha *}^\beta : H_n(X_\beta; G) \to H_n(X_\alpha; G)$ and $p_\alpha^{\beta *} : H_n(X_\alpha; G) \to H_n(X_\beta; G)$, which form inverse systems

$$\{H_n(A_\alpha;G), p^\beta_{\alpha*}, \mathrm{cov}_\infty(X,A)\} \text{ and } \{H_n(X_\alpha;G), p^\beta_{\alpha*}, \mathrm{cov}_\infty(X,A)\}$$

and direct systems

$$\{H^n(A_\alpha;G), p^{\beta*}_\alpha, \mathrm{cov}_\infty(X,A)\} \text{ and } \{H^n(X_\alpha;G), p^{\beta*}_\alpha, \mathrm{cov}_\infty(X,A)\}.$$

Let

$$\check{H}^\infty_n(A;G)_{(X,A)} = \varprojlim \{H_n(A_\alpha;G), p^\beta_{\alpha*}, \mathrm{cov}_\infty(X,A)\},$$

$$\check{H}^\infty_n(X;G)_{(X,A)} = \varprojlim \{H_n(X_\alpha;G), p^\beta_{\alpha*}, \mathrm{cov}_\infty(X,A)\},$$

$$\hat{H}^n_\infty(A;G)^{(X,A)} = \varinjlim \{H^n(A_\alpha;G), p^{\beta*}_\alpha, \mathrm{cov}_\infty(X,A)\},$$

$$\hat{H}^n_\infty(X;G)^{(X,A)} = \varinjlim \{H^n(X_\alpha;G), p^{\beta*}_\alpha, \mathrm{cov}_\infty(X,A)\}.$$

Our main aim is to show that the groups $\check{H}^\infty_n(A;G)$ and $\hat{H}^\infty_n(A;G)_{(X,A)}$, $\hat{H}^n_\infty(A;G)$ and $\hat{H}^n_\infty(A;G)^{(X,A)}$, $\check{H}_n(X;G)$ and $\check{H}_n(X;G)_{(X,A)}$, $\hat{H}^n(X;G)$ and $\hat{H}^n(X;G)^{(X,A)}$ are isomorphical groups.

Next we define a function $\varphi : \mathrm{cov}_\infty(X,A) \to \mathrm{cov}_\infty(A,\emptyset)$. Let $\alpha = \{\alpha_v\}_{v\in(V_\alpha,V^A_\alpha)} \in \mathrm{cov}_\infty(X,A)$. Assume that $(\varphi(\alpha))_v = \alpha_v \cap A$ for $v \in V^A_\alpha$. We have defined the border cover $\varphi(\alpha) \in \mathrm{cov}_\infty(A,\emptyset)$ indexed by pair $(V_\alpha,\emptyset)$.

Let $K_\alpha = X \setminus \bigcup\limits_{v\in V_\alpha} \alpha_v$. Note that

$$A \setminus (K_\alpha \cap A) = \bigcup_{v\in V^A_\alpha} (\alpha_v \cap A) = \bigcup_{v\in V^A_\alpha} (\varphi(\alpha))_v.$$

It is clear that $K_\alpha \cap A$ is a compact subset of the subspace A. Thus, $\varphi(\alpha) \in \mathrm{cov}_\infty(A,\emptyset)$. The defined function is an order preserving function.

It is easy to show that the image of function φ is a cofinal subset of set $\mathrm{cov}_\infty(A,\emptyset)$. Note that $A_\alpha = A_{\varphi(\alpha)}$. By $\varphi_\alpha : A_{\varphi(\alpha)} \to A_\alpha$ denote this simplicial isomorphism. Hence, the family of pairs (φ_α,φ) induces a map of inverse systems and direct systems

$$(\varphi_{\alpha*},\varphi) : \{H_n(A_\alpha;G), p^\beta_{\alpha*}, \mathrm{cov}_\infty(A,\emptyset)\} \to \{H_n(A_\alpha;G), p^\beta_{\alpha*}, \mathrm{cov}_\infty(X,A)\}$$

and

$$(\varphi^*_\alpha,\varphi) : \{H^n(A_\alpha;G), p^{\beta*}_\alpha, \mathrm{cov}_\infty(X,A)\} \to \{H_n(A_\alpha;G), p^{\beta*}_\alpha, \mathrm{cov}_\infty(A,\emptyset)\}.$$

Let $\Phi_n = \varprojlim (\varphi_{\alpha*},\varphi)$ and $\Phi^n = \varinjlim (\varphi^*_\alpha,\varphi)$. Since all homomorphisms $\varphi_{\alpha*}$ and φ^*_α are isomorphisms, the limit homomorphisms

$$\Phi_n : \check{H}^\infty_n(A;G) \to \check{H}^\infty_n(A;G)_{(X,A)}$$

and

$$\Phi^n : \hat{H}^n_\infty(A;G)^{(X,A)} \to \hat{H}^n_\infty(A;G)$$

are isomorphisms.

Lets us also define a function $\psi : \mathrm{cov}_\infty(X,A) \to \mathrm{cov}_\infty(X,\emptyset)$. For each $\alpha = \{\alpha_v\}_{v\in(V_\alpha,V_\alpha^A)} \in \mathrm{cov}_\infty(X,A)$ assume that $(\psi(\alpha))_v = \alpha_v$, $v \in V_\alpha$. The family $\psi(\alpha)$ is indexed by $(V_\alpha,\emptyset)$ and $\psi(\alpha) \in \mathrm{cov}_\infty(X,\emptyset)$.

Note that $X_\alpha = X_{\psi(\alpha)}$. Let $\psi_\alpha : X_{\psi(\alpha)} \to X_\alpha$ be a simplicial isomorphism. The family of pairs (ψ_α,ψ) induce the maps of inverse and direct systems

$$(\psi_{\alpha *},\psi) : \{H_n(X_\alpha;G), p^\beta_{\alpha *}, \mathrm{cov}_\infty(X,\emptyset)\} \to \{H_n(X_\alpha;G), p^\beta_{\alpha *}, \mathrm{cov}_\infty(X,A)\}$$

and

$$(\psi^*_\alpha,\psi) : \{H^n(X_\alpha;G), p^{\beta *}_\alpha, \mathrm{cov}_\infty(X,A)\} \to \{H^n(X_\alpha;G), p^{\beta *}_\alpha, \mathrm{cov}_\infty(X,\emptyset)\}.$$

Let $\Psi_n = \varprojlim(\psi_{\alpha *},\psi)$ and $\Psi^n = \varinjlim(\psi^*_\alpha,\psi)$. Since each $\psi_{\alpha *}$ and ψ^*_α are isomorphisms, the induced limit homomorphisms

$$\Psi_n : \check{H}^\infty_n(X;G) \to \check{H}^\infty_n(X;G)_{(X,A)}$$

and

$$\Psi^n : \hat{H}^n_\infty(X;G)^{(X,A)} \to \hat{H}^n_\infty(X;G)$$

are isomorphisms.

There exist the limit sequences

$$\cdots \longleftarrow \check{H}^\infty_n(X,A;G) \xleftarrow{j'^\infty_n} \check{H}^\infty_n(X;G)_{(X,A)} \xleftarrow{i'^\infty_n} \check{H}^\infty_n(A;G)_{(X,A)} \xleftarrow{\partial'^\infty_{n+1}} \check{H}^\infty_{n+1}(X,A;G) \longleftarrow \cdots$$

and

$$\cdots \longrightarrow \hat{H}^n_\infty(X,A;G) \xrightarrow{j'^n_\infty} \hat{H}^n_\infty(X;G)^{(X,A)} \xrightarrow{i'^n_\infty} \hat{H}^n_\infty(A;G)^{(X,A)} \xrightarrow{\delta'^n_\infty} \hat{H}^{n+1}_\infty(X,A;G) \longrightarrow \cdots$$

generated by the families consisting of homology and cohomology sequences of simplicial pairs $(X_\alpha,A_\alpha), \alpha \in \mathrm{cov}_\infty(X,A)$, respectively.

Consider the diagrams

$$\check{H}^\infty_n(X,A;G) \xrightarrow{\partial'^\infty_n} \check{H}^\infty_{n-1}(A;G)_{(X,A)} \xleftarrow{\Phi_{n-1}} \check{H}^\infty_{n-1}(A;G)$$

and

$$\hat{H}^n_\infty(A;G) \xleftarrow{\Psi^n} \hat{H}^n_\infty(A;G)^{(X,A)} \xrightarrow{\delta'^n_\infty} \hat{H}^{n+1}_\infty(X,A;G)$$

and define the boundary homomorphism of Čech border homology groups and coboundary homomorphism of Čech border cohomology groups as compositions

$$\partial^\infty_n = (\Phi_{n-1})^{-1} \cdot \partial'^\infty_n$$

and

$$\delta^n_\infty = {\delta'}^n_\infty \cdot (\Psi^n)^{-1}.$$

In this way we arrive to the following theorems.

Theorem 1.5. *Let $f : (X,A) \to (Y,B)$ be a proper map. Then hold the following equalities*

$$(f_{|A})^\infty_* \cdot \partial^\infty_n = \partial^\infty_n \cdot f^\infty_*$$

and

$$\delta^{n-1}_\infty (f_{|A})^*_\infty = f^*_\infty \cdot \delta^{n-1}_\infty.$$

Let $i : A \to X$ and $j : X \to (X,A)$ be the inclusion maps.

Theorem 1.6. *Let $(X,A) \in ob(\mathscr{N}^2_p)$. Then the Čech border cohomology sequence*

$$\cdots \longrightarrow \check{H}^{n-1}_\infty(A;G) \xrightarrow{\delta^{n-1}_\infty} \check{H}^n_\infty(X,A;G) \xrightarrow{j^*_\infty} \check{H}^n_\infty(X;G) \xrightarrow{i^*_\infty} \check{H}^n_\infty(A;G) \longrightarrow \cdots$$

is exact while the Čech border homology sequence

$$\cdots \longleftarrow \hat{H}^\infty_{n-1}(A;G) \xleftarrow{\partial^\infty_n} \hat{H}^\infty_n(X,A;G) \xleftarrow{j^\infty_*} \hat{H}^\infty_n(X;G) \xleftarrow{i^\infty_*} \hat{H}^\infty_n(A;G) \longleftarrow \cdots$$

is partially exact.

Theorem 1.7. *Let $(X,A) \in ob(\mathscr{N}^2_p)$ and G be an abelian group. If U is open in X and $\bar{U} \subset \mathrm{int}A$, then the inclusion map $i : (X \setminus U, A \setminus U) \to (X,A)$ induces isomorphisms*

$$i^\infty_* : \check{H}^\infty_n(X \setminus U, A \setminus U) \to \check{H}^\infty_n(X,A;G)$$

and

$$j^*_\infty : \hat{H}^n_\infty(X,A;G) \to \hat{H}^n_\infty(X \setminus U, A \setminus U)$$

Theorem 1.8. *If X is a compact space, then for each $n \neq 0$,*

$$\check{H}^\infty_n(X;G) = 0 = \check{H}^n_\infty(X;G)$$

and

$$\hat{H}^\infty_0(X;G) = G = \hat{H}^0_\infty(X;G).$$

Thus, Čech border homology (cohomology) functors $\check{H}^\infty_n(-,-;G)(\hat{H}^n_\infty(-,-;G)) : \mathscr{N}^2_p \to \mathscr{A}b$ satisfy the Steenrod-Eilenberg type axioms (cf.[E-St]): axiom of natural transformation, axiom of partially exactness (axiom of exactness), axiom of excision and axiom of dimension; but they do not satisfy the proper homotopy axiom.

The above obtained results yield the next theorem.

Theorem 1.9. *Let (X, A, B) be a triple of normal space X and its closed subsets A and B with $B \subset A$. Then the Čech border homology sequence*

$$\cdots \leftarrow \check{H}^{\infty}_{n-1}(A, B; G) \xleftarrow{\bar{\partial}^{\infty}_{n}} \check{H}^{\infty}_{n}(X, A; G) \xleftarrow{\bar{j}^{\infty}_{*}} \check{H}^{\infty}_{n}(X, B; G) \xleftarrow{\bar{i}^{\infty}_{*}} \check{H}^{\infty}_{n}(A, B; G) \leftarrow \cdots$$

and the Čech border cohomology sequence

$$\cdots \rightarrow \hat{H}^{n-1}_{\infty}(A, B; G) \xrightarrow{\bar{\delta}^{n}_{\infty}} \hat{H}^{n}_{\infty}(X, A; G) \xrightarrow{\bar{j}^{*}_{\infty}} \hat{H}^{n}_{\infty}(X, B; G) \xrightarrow{\bar{i}^{*}_{\infty}} \hat{H}^{n}_{\infty}(A, B; G) \rightarrow \cdots$$

are partially exact and exact, respectively. Here $\bar{\partial}^{\infty}_{n} = j'^{\infty}_{n-1} \cdot \partial^{\infty}_{n}$, $\bar{\delta}^{n}_{\infty} = \delta^{n}_{\infty} \cdot j'^{n-1}_{\infty}$ and $\bar{j}^{*}_{\infty}$ and $\bar{i}^{*}_{\infty}$ are the homomorphisms induced by the inclusion maps $j' : A \to (A, B)$, $\bar{i} : (A, B) \to (X, B)$ and $\bar{j} : (X, B) \to (X, A)$.

2 On some applications of Čech border homology and cohomology groups

Now we are mainly interested in the following problem: how to characterize the Čech homology and cohomology groups, coefficients of cyclicity, and cohomological dimensions of remainders of Stone-Čech compactifications of spaces.

Our main result about the connection between Čech (co)homology groups of remainders and Čech border (co)homology groups of spaces is.

Theorem 2.1. *Let $(X, A) \in ob(\mathscr{M}^2_p)$ and let $(\beta X, \beta A)$ be the pair of Stone-Čech compactifications of X and A. Then*

$$\check{H}^{f}_{n}(\beta X \setminus X, \beta A \setminus A; G) = \check{H}^{\infty}_{n}(X, A; G)$$

and

$$\hat{H}^{n}_{f}(\beta X \setminus X, \beta A \setminus A; G) = \hat{H}^{n}_{\infty}(X, A; G).$$

The theory of cohomological dimension has become an important branch of dimension theory since A. Dranishnikov solved P.S. Alexandrov's problem and developed the theory of extension dimension ([D], [D-Dy]).

Our next aim is to study some questions of theory of cohomological dimension. In particular, we now give a description of cohomological dimension of remainder of Stone-Čech compactification of metrizable space.

Following Y. Kodama (see the appendix of [N]) and T. Miyata [Mi] we give the following definition.

Definition 2.2. Let $n \geq 0$. The border small cohomological dimension of normal space X with respect to abelian group G, $d^{f}_{\infty}(X; G) \leq n$ if for every $m \geq n$ and closed subset A of X the homomorphism $i^{*}_{A,\infty} : \hat{H}^{m}_{\infty}(X; G) \to \hat{\mathrm{H}}^{m}_{\infty}(A; G)$ induced by the inclusion $i : A \to X$ is an epimorphism.

We say $d^f_\infty(X;G) = n$ if $d^f_\infty(X;G) \leq n$ but not $d^f_\infty(X;G) \leq n-1$.

Note that $d^f_\infty(X;G) = +\infty$ if the inequality $d^f_\infty(X;G) \leq n$ does not hold for any n.

The border small cohomological dimension of X with coefficient group G is a function $d^f_\infty : \mathscr{N} \to \mathrm{N} \cup \{0, +\infty\} : X \to n$, where $d^f_\infty(X;G) = n$ and N is the set of all positive integers.

Theorem 2.3. *Let X be a metrizable space. Then the following equality*

$$d^f_\infty(X;G) = d_f(\beta X \setminus X; G)$$

holds, where $d_f(\beta X \setminus X; G)$ is the small cohomological dimension of $\beta X \setminus X$ (see [N], p.199*).*

Theorem 2.4. *Let A be a closed subspace of a normal space X. Then*

$$d^f_\infty(A;G) \leq d^f_\infty(X;G).$$

Corollary 2.5. *For each closed subspace A of a metrizable space X*

$$d^f_\infty(A;G) \leq d_f(\beta X \setminus X; G).$$

Definition 2.6. Let $n \geq 0$. The border large cohomological dimension of normal space X with respect to abelian group G, $D^f_\infty(X;G) \leq n$ if for every $m \geq n+1$ and closed subset A of X, $\hat{H}^m_\infty(X, A; G) = 0$

We say $D^f_\infty(X;G) = n$ if $D^f_\infty(X;G) \leq n$ but not $D^f_\infty(X;G) \leq n-1$.

Note that $D^f_\infty(X;G) = +\infty$ if the inequality $D^f_\infty(X;G) \leq n$ does not hold for any n.

The border large cohomological dimension of X with coefficient group G is a function $D^f_\infty : \mathscr{N} \to \mathrm{N} \cup \{0, +\infty\} : X \to n$, where $D^f_\infty(X;G) = n$ and N is the set of all positive integers.

Theorem 2.7. *For each metrizable space X, one has*

$$D^f_\infty(X;G) = D_f(\beta X \setminus X; G),$$

where $D_f(\beta X \setminus X; G)$ is the large cohomological dimension of $\beta X \setminus X$ (see [N], p.199*).*

Theorem 2.8. *If A is a closed subset of normal space X, then*

$$D^f_\infty(A;G) \leq D^f_\infty(X;G).$$

Corollary 2.9. *For each closed subspace A of metrizable space X, one has*

$$D^f_\infty(A;G) \leq D_f(\beta X \setminus X; G).$$

Theorem 2.10. *If X is a normal space, then*

$$d^f_\infty(X;G) \leq D^f_\infty(X;G).$$

Corollary 2.11. *For each metrizable space X, one has*

$$d_f(\beta X \setminus X; G) \leq D^f_\infty(X; G)$$

and

$$d^f_\infty(X; G) \leq D_f(\beta X \setminus X; G).$$

Remark 2.12. The results of this paper also hold for spaces satisfying the compact axiom of countability. Recall that a space X satisfies the compact axiom of countability if for each compact subset $B \subset X$ there exists a compact subset $B^{'} \subset X$ such that $B \subset B^{'}$ and $B^{'}$ has a countable or finite fundamental systems of neighbourhoods (see Definition 4 of [Sm$_4$], p.143). A space X is complete in the sense of Čech if and only if it is G_δ type set in some compact extension. Each locally metrizable spaces, complete in the seance of Čech spaces [Č] and locally compact spaces satisfy the compact axiom of countability.

References

[A] J. M. Aarts, Completeness degree. A generalization of dimension. Fund. Math. 63(1968), 27-41.

[A-N] J. M. Aarts and T. Nishiura, Dimension and Extensions. North-Holland Math. Library, Amsterdam, London, New York and Tokyo, 1993.

[Ak-Chin-T] Y. Akaike, N. Chinen and K. Tomoyasu, Large inductive dimension of the Smirnov remainder, Houston J. Math., 34 (2008), 501-510.

[Al] P. Alexandroff, On the dimension of normal spaces. Proc. Roy. London. Ser. A., 189(1947), 11-39.

[B$_1$] V. Baladze, On some combinatorial properties of remainders of extensions of topological spaces and on factorization of uniform mappings. Soobshch. Akad. Nauk Gruzin. SSR., 98(1980), 293-296.

[B$_2$] V. Baladze, On Uniform shapes. Bull. Georgian Acad. Sci., 169(2002), 26-29.

[B$_3$] V. Baladze, Characterization of precompact shape and homology properties of remainders. Topology Appl. 142(2004), 181-196.

[B-Tu] V. Baladze and L.Turmanidze, On homology and cohomology groups of remainders. Georgian Math. J. 11(2004), 613-633.

[Ba] B. J. Ball, Geometric topology and shape theory: a survey of problems and results, Bull. Amer. Math.Soc. 82(1976), 791-804.

[C] A. Calder, The cohomotopy groups of Stone-Čech increments, Indag. Math. 34 (1972), 37-44.

[Ch] M. G. Charalambous, Spaces with increment of dimension n. Fund. Math. 93(1976), 97-107.

[Chi_1] A. Chigogidze, Inductive dimensions for completely regular spaces. Comment. Math. Univ. Carolinae, 18(1977), 623-637.

[Chi_2] A. Chigogidze, On the dimension of increments of Tychonoff spaces. Fund. Math.,111(1981), 25-36.

[Č] E. Čech, On bicompact spaces. Ann. Math., 38 (1937), 823-844.

[D] A.N. Dranishnikov, On a problem of P.S. Alexandrov, Math. USSR Sbornik., 63(1989), 539-545.

[D-Dy] A. Dranishnikov and J. Dydak, Extension dimension and extension types, Proc. Steklov Inst. Math., 212 (1996), 55-88

[E-St] S. Eilenberg and N. E. Steenrod, Foundations of Algebraic Topology. Princeton University Press, Princeton, New Jersey, 1952.

[En] R. Engelking, General Topology. PWN-Polish Scientific Publishers, Warsaw, 1977.

[F_1] H. Freudenthal, Neuaufbau der Endentheorie, Ann. Math., 43 (1942), 261-279.

[F_2] H. Freudenthal, Kompaktisierungen und Bikompaktisierungen. Ind. Math. Amst., 13(1951), 184-192.

[I_1] H. Inasaridze, On finite order extensions and remainders of completely regular spaces, Doklady Acad. Nauk USSR 166(1966),1043-1045.

[I_2] H. Inasaridze, A generalization of perfect mappings, Doklady Acad. Nauk USSR 168 (1966), 266-268.

[I_3] H. Inasaridze, Universal functors, Bull. Georgian Acad. Sci., 38(1965), 513-520.

[K_1] J. Keesling, The Stone-Čech compactification and shape dimension. Topology Proceedings, 2(1977), 483-508.

[K_2] J. Keesling, Decompositions of the Stone-Čech compactification which are shape equivalences. Pac. J.Math., 75(1978), 455-466.

[K-Sh] J. Keesling and R.B. Sher, Shape properties of the Stone-Čech compactification. General Topology and Appl.,9(1978), 1-8.

[M-S] S. Mardešić and J. Segal, Shape theory. North-Holland Publishing Co., Amsterdam-New York, 1982.

[Mi] T. Miyata, Cohomological Dimension of Uniform Spaces, Quaestiones Mathematicae, 19(1996), 137-162.

[Mo] K. Morita, On bicompactifications of semibicompact spaces. Sci. Rep. Tokyo Bunrika Daigaku. Sect. A., 4(1952), 222-229.

[N] K. Nagami, Dimension Theory (with an appendix: Y. Kodama, Cohomological Dimension Theory). Academic Press, New York and London, 1970.

[Sk] E.G. Skljarenko, Some questions in the theory of bicompactifications.Izv. Akad. Nauk SSSR Ser. Mat., 26(1962), 427-452.

[Sm_1] Ju. M. Smirnov, On the dimension of proximity spaces. Mat. Sb. (N.S.) 38(80)(1956), 283-302.

[Sm_2] Ju. M. Smirnov, Proximity and construction of compactifications with given properties. Proc. of the Second Prague Topol. Symp. 1966, Prague, (1967), 332-340.

[Sm_3] Ju. M. Smirnov, Dimension of increments of proximity spaces and of topological spaces, Dokl. Akad. Nauk SSSR., 168(1966), 528-531.

[Sm_4] Ju. M. Smirnov, On the dimension of remainders in bicompact extensions of proximity and topological spaces, Mat. Sb. (N.S.) 69(111)(1966), 141-160.

[Sm_5] Ju. M. Smirnov, On the dimension of remainders in bicompact extensions of proximity and topological spaces. II, Mat. Sb. (N.S.) 71 (113)(1966), 454-482.

[Ts] I.Tsereteli, Two analogues of Hurewicz's theorem, Proceedings of Tbilisi university, 264, Mathematics. Mechanics. Astronomy. 21, 73-90.

[Z_1] L. Zambakhidze, On some properties of spaces with compact remainders of finite orders, Dokl. Akad. Nauk SSSR, 192(1970), 263-266.

[Z_2] L. Zambakhidze, On dimension-like functions, Proc. Tbilisi Math. Inst., 56 (1977), 52-98.

[V] H. de Vries, Compact spaces and compactifications. Doct. Diss, Amsterdam, 1962.