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Dense Subsets of Function Spaces with No Non-Trivial Convergent Sequences

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A classical theorem of Gerlits, Nagy and Pytkeev proved in 1982 states that for any Tychonoff space X, if $C_p(X)$ is a k-space, then it is Fréchet–Urysohn. Even if $C_p(X)$ is assumed to be sequential, it is not easy at all to prove that it has the Fréchet–Urysohn property.

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This theorem cannot be generalized to arbitrary subspaces of the spaces $C_p(X)$. Indeed, $\beta\omega$ is a compact subspace of $C_p(C_p(\beta\omega))$ but it does not even have countable tightness.

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The Arhangel'skii–Franklin sequential space F whose discovery dates back to 1968, embeds in $\mathbb{R}^{\mathfrak{c}} = C_p(D(\mathfrak{c}))$; here $D(\mathfrak{c})$ is the discrete space of cardinality $\mathfrak{c}$. Since the space F is not Fréchet–Urysohn, even sequentiality of a subspace of $C_p(X)$ does not imply its Fréchet–Urysohn property.

Now, if K is compact, then the k-property of a subspace of $C_p(K)$ is equivalent to its Fréchet–Urysohn property: this was proved by Pytkeev in 1982. In this paper we will show that the same equivalence takes place if K is σ-pseudocompact.

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Pytkeev established in 1993 that a compact space X is scattered if and only if $C_p(X)$ has a dense Fréchet–Urysohn subspace. This implies that $C_p(X)$ has a dense k-subspace if and only if it is Fréchet–Urysohn.

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We will show that this result cannot be generalized to countably compact spaces X, i.e., the existence of a dense Fréchet–Urysohn subspace in $C_p(X)$ does not imply that $C_p(X)$ is Fréchet–Urysohn. However, if a countably compact space X is sequential and $C_p(X)$ has a dense k-subspace, then X is scattered.

The above-mentioned result of Pytkeev implies that for a non-scattered compact space X, no dense subspace of $C_p(X)$ has the Fréchet–Urysohn property. We will establish that a Corson compact space X is scattered or, equivalently, $C_p(X)$ is Fréchet–Urysohn if and only if every dense subspace of $C_p(X)$ has a non-trivial convergent sequence. If a σ-compact space X is uncountable and $w(X) = \omega$, then any dense subspace of $C_p(X)$ has a dense subset without non-trivial convergent sequences.

In 2000, Alas, Sanchis, Tkachenko, Tkachuk and Wilson proved that, under the Booth lemma, $[0,1]^{\mathfrak{c}}$ has a countable dense submaximal subspace and, in particular, it has a countable dense subspace without non-trivial convergent sequences.

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In this paper we will construct in ZFC such a countable dense subspace in $\mathbb{R}^{\mathfrak{c}}$. Besides, we will show that, for any cardinal $\kappa \geqslant \mathfrak{c}$, the space $\mathbb{R}^{\kappa}$ has a dense subspace without non-trivial convergent sequences.

1. Theorem.

Given a σ-bounded space X, if $Y \subset C_p(X)$ is a k-space, then Y has the Fréchet–Urysohn property.

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Proof.

Observe that $K = \upsilon X$ is σ-compact and the restriction map $\pi : C_p(K) \to C_p(X)$ is a bijection. If a set $L \subset C_p(X)$ is compact, then $\pi^{-1}(L) \subset C_p(K)$ is homeomorphic to L so L is Fréchet–Urysohn being Gul'ko compact.

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Therefore the k-property of Y implies that Y is sequential and hence $t(Y) \leqslant \omega$. Consider the space $Z = \pi^{-1}(Y) \subset C_p(K)$ and assume that a set $A \subset Z$ is not closed in Z. For the set $B = \pi(A)$ it follows from $A = \pi^{-1}(B)$ that B is not closed in Y. Let $S \subset B$ be a sequence converging to a point $x \in Y \backslash B$.

The restriction of the map π to the countable set $\pi^{-1}(\{x\} \cup S)$ is a homeomorphism so the sequence $\pi^{-1}(S) \subset A$ converges to the point $\pi^{-1}(x) \in Z \setminus A$; this proves that Z is also sequential.

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The space $C_p(K)$ has the Whyburn property and therefore Z is also a Whyburn space. This, together with sequentiality of Z easily implies that Z has the Fréchet–Urysohn property.

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The space $C_p(K)$ has the Whyburn property and therefore Z is also a Whyburn space. This, together with sequentiality of Z easily implies that Z has the Fréchet–Urysohn property.

Fix any set $E \subset Y$ and a point $x \in \mathrm{cl}_Y(E)$. There exists a countable set $H \subset E$ such that $x \in \mathrm{cl}_Y(H)$. If $G = \pi^{-1}(E)$ and $D = \pi^{-1}(H)$, then the point $y = \pi^{-1}(x)$ belongs to the closure in Z of the set $D \subset G$ because the restriction of the map π to the set $\{y\} \cup D$ is a homeomorphism. Since Z is Fréchet–Urysohn, we can find a sequence $T \subset G$ that converges to y. Then the sequence $\pi(T) \subset E$ converges to x and witnesses that Y has the Fréchet–Urysohn property.

3. Corollary.

For any σ-pseudocompact X, if $Y \subset C_p(X)$ is a k-space, then Y has the Fréchet–Urysohn property.

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For any σ-pseudocompact X, if $Y \subset C_p(X)$ is a k-space, then Y has the Fréchet–Urysohn property.

4. Example.

The Arhangel'skii–Franklin space S is countable, sequential but not Fréchet–Urysohn. Since S embeds in $C_p(C_p(S))$ and $w(C_p(S)) \leqslant \omega$, we can see that a sequential subspace of $C_p(X)$ need not be Fréchet–Urysohn even if $X = C_p(S)$ is second countable.

5. Definition.

Say that a space X has the Banakh property, if there exists a sequence $\{F_n : n \in \omega\}$ of closed nowhere dense subsets of X that swallows all compact subsets of X, i.e., for any compact set $K \subset X$, there exists $n \in \omega$ such that $K \subset F_n$.

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Krupski and Marciszewski established that a space with the Banakh property can be sequential but cannot be Fréchet–Urysohn. The following is immediate from the definition.

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Krupski and Marciszewski established that a space with the Banakh property can be sequential but cannot be Fréchet–Urysohn. The following is immediate from the definition.

6. Proposition.

If a space X has the Banakh property, then any dense subset of X also has the Banakh property.

Krupski and Marciszewski also proved that a compact space X is scattered if and only if $C_p(X)$ fails to have the Banakh property. We will present a version of this result for countably compact spaces.

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7. Proposition.

Suppose that a space X has a subspace Y with the Baire property in which we can find a countable π-network consisting of infinite subsets of Y. Then $C_p(X)$ has the Banakh property.

Proof.

Let $\mathcal{P}$ be a countable π-network in Y such that every element of $\mathcal{P}$ is infinite. Therefore the set $H(P,n) = \{f \in C_p(X) : f(P) \subset [-n,n]\}$ is closed and nowhere dense in $C_p(X)$ for all $P \in \mathcal{P}$ and $n \in \mathbb{N}$. To see that the family $\mathcal{N} = \{H(P,n) : P \in \mathcal{P},\ n \in \mathbb{N}\}$ witnesses the Banakh property of $C_p(X)$, fix any compact subset $K \subset C_p(X)$. For each $x \in X$ there exists a number $n(x) \in \mathbb{N}$ such that $|f(x)| \leqslant n(x)$ for every $f \in K$.

Proof.

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It is easy to see that the set $Q_n = \{x \in X : n(x) \leqslant n\}$ is closed in X for every $n \in \mathbb{N}$. Since $X = \bigcup_{n\in\mathbb{N}} Q_n$, it follows from the Baire property of Y that $Q_n \cap Y$ has non-empty interior in Y for some $n \in \mathbb{N}$. Pick $P \in \mathcal{P}$ such that $P \subset Q_n \cap Y$ and observe that $K \subset H(P,n)$; this proves that the family $\mathcal{N}$ swallows all compact subsets of $C_p(X)$.

8. Corollary.

If X is a non-empty space, then $C_p(C_p(X))$ has the Banakh property.

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Proof.

Just observe that $\mathbb{R}$ embeds in $C_p(X)$ and apply Proposition 7.

9. Theorem.

Let X be a countably compact sequential space. If X is not scattered, then $C_p(X)$ has the Banakh property.

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Let X be a countably compact sequential space. If X is not scattered, then $C_p(X)$ has the Banakh property.

Proof.

If X is not scattered, then take a crowded subspaces $Z \subset X$. It easily follows from $t(Z) \leqslant \omega$ that we can find a crowded countable subspace $A \subset Z$; then the set $Y = \overline{A}$ has the Baire property being countably compact. It follows from sequentiality of Y that for every $y \in Y$ we can find a countable π-network $\mathcal{P}_y$ at y consisting of infinite subsets of A. Therefore $\mathcal{P} = \bigcup\{\mathcal{P}_a : a \in A\}$ is a countable π-network in Y consisting of infinite subsets of Y. This shows that Proposition 7 is applicable to conclude that $C_p(X)$ has the Banakh property.

10. Corollary.

If X is a countably compact sequential space and $C_p(X)$ has a dense k-subspace, then X is scattered.

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Proof.

Let $D \subset C_p(X)$ be a dense k-subspace of $C_p(X)$. Note first that D must be Fréchet–Urysohn by Theorem 1. If X is not scattered, then $C_p(X)$ has the Banakh property by Theorem 9. Therefore D also has the Banakh property by Proposition 6. However, a Fréchet–Urysohn space cannot have the Banakh property by a result of Krupski and Marciszewski. This contradiction shows that X is scattered.

If X is a compact space and $C_p(X)$ has a dense k-subspace, then $C_p(X)$ is Fréchet–Urysohn: this was proved by Pytkeev in 1993. The following example shows that we cannot expect that for countably compact spaces.

If X is a compact space and $C_p(X)$ has a dense k-subspace, then $C_p(X)$ is Fréchet–Urysohn: this was proved by Pytkeev in 1993. The following example shows that we cannot expect that for countably compact spaces.

11. Example.

If X is the ordinal ω_1 with its interval topology, then X is a countably compact space for which $C_p(X)$ has a dense Fréchet–Urysohn subspace while $t(C_p(X)) > \omega$.

Proof.

The ordinal $\omega_1 + 1$ is a scattered compact space. Let Y be its quotient space obtained by identifying the points 0 and ω_1. Then Y is also a scattered compact space and hence $C_p(Y)$ has the Fréchet–Urysohn property. If $t \in Y$ is the point represented by the set $\{0, \omega_1\}$ then let $\varphi(0) = t$ and $\varphi(\alpha) = \alpha$ for any $\alpha \in \omega_1 \setminus \{0\}$. Then $\varphi : \omega_1 \to Y$ is easily seen to be a condensation and hence $C_p(\omega_1)$ has a dense subspace homeomorphic to the Fréchet–Urysohn space $C_p(Y)$. Finally observe that $t(C_p(X)) > \omega$ because X is not Lindelöf.

12. Proposition.

Suppose that X and Y are crowded spaces for which

(a) there exists a dense set $A = \{a_n : n \in \omega\} \subset X$ without non-trivial convergent sequences;

(b) there exists a sequence $\{D_n : n \in \omega\}$ of discrete subspaces of Y such that $D_n \subset D_{n+1}$ for any $n \in \omega$ and $D = \bigcup_{n\in\omega} D_n$ is dense in Y.

Then the set $E = \bigcup\{\{a_n\} \times D_n : n \in \omega\}$ is dense in $X \times Y$ and has no non-trivial convergent sequences.

Proof.

Given any sets $U \in \tau^*(X)$ and $V \in \tau^*(Y)$, there exists $n \in \omega$ such that $D_n \cap V \neq \emptyset$. The set U being infinite, we can choose $m \geqslant n$ such that $a_m \in U$. Pick a point $d \in D_n \cap V$ and observe that the point $(a_m, d) \in D \cap (U \times V)$ witnesses that E is dense in $X \times Y$.

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Now, assume that a sequence $S \subset E$ converges to a point $z = (a_n, d) \in E \backslash S$. The set D_n being discrete, the intersection $G = S \cap (\{a_n\} \times D_n)$ must be finite and hence the sequence $S \backslash G$ still converges to z. If $p : X \times Y \to X$ is the projection, then the sequence $S' = p(S \backslash G) \subset A$ converges to the point $a_n \in A \backslash S'$ which is a contradiction. Therefore E has no non-trivial convergent sequences.

13. Corollary.

If crowded space X has a countable dense subspace without non-trivial convergent sequences and a crowded space Y is separable, then $X \times Y$ has a dense subspace with no non-trivial convergent sequences.

The proof of the following statement a straightforward modification of the proof of Lemma 3.4 from the paper

V.V. Tkachuk, *Discrete reflexivity in function spaces,* Bull. Belg. Math. Soc., **22:1**(2015), 1-14

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14. Lemma.

Suppose that F *is a compact subset of an infinite space* X *and denote by* I_F *the set* $\{f \in C_p(X) : f(F) \subset \{0\}\}$. *Assume additionally, that there is a discrete set* $D \subset (X \backslash F) \times (X \backslash F)$ *such that* $|D| = iw(X)$. *Then there exists a discrete set* $\Omega \subset (C_p(X, [-2, 2]) \cap I_F) \backslash C_p(X, \mathbb{I})$ *such that* $|\Omega| \leqslant iw(X)$ *and* $C_p(X, \mathbb{I}) \cap I_F \subset \overline{\Omega}$.

15. Theorem.

Suppose that X is a second countable uncountable σ-compact space and D is a dense subset of $C_p(X)$. Then there exists a set $E \subset D$ such that $\overline{E} = C_p(X)$ and E has no non-trivial convergent sequences.

Proof.

There is a family $\{K_n : n \in \omega\}$ of compact subsets of X such that $X = \bigcup_{n\in\omega} K_n$. All sets K_n cannot be scattered so $C_p(X)$ has the Banakh property by a theorem of Krupski and Maciszewski and hence we can fix an increasing family $\mathcal{F} = \{F_n : n \in \omega\}$ of nowhere dense closed subsets of $C_p(X)$ such that every compact subspace of $C_p(X)$ is contained in a member of $\mathcal{F}$.

Proof.

There is a family $\{K_n : n \in \omega\}$ of compact subsets of X such that $X = \bigcup_{n\in\omega} K_n$. All sets K_n cannot be scattered so $C_p(X)$ has the Banakh property by a theorem of Krupski and Maciszewski and hence we can fix an increasing family $\mathcal{F} = \{F_n : n \in \omega\}$ of nowhere dense closed subsets of $C_p(X)$ such that every compact subspace of $C_p(X)$ is contained in a member of $\mathcal{F}$.

The space $C_p(X)$ is selectively separable so we can choose a finite set $E_n \subset D\backslash F_n$ for every $n \in \omega$ in such a way that $E = \bigcup_{n\in\omega} E_n$ is dense in $C_p(X)$. If K is a compact subset of E, then there exists $n \in \omega$ for which $K \subset F_n$. Our choice of E shows that $K \subset E_0 \cup \ldots \cup E_{n-1}$, i.e., K is finite. Therefore all compact subsets of E are finite so it has no non-trivial convergent sequences as promised.

16. Corollary.

There is a countable dense subset D in the space $\mathbb{R}^{\mathfrak{c}}$ that has no non-trivial convergent sequences.

16. Corollary.

There is a countable dense subset D in the space $\mathbb{R}^{\mathfrak{c}}$ that has no non-trivial convergent sequences.

Proof.

Give the ordinal $\mathfrak{c}$ the discrete topology and let $\varphi : \mathfrak{c} \to \mathbb{I}$ be a bijection. For any $f \in C_p(\mathbb{I})$ let $\varphi^*(f) = f \circ \varphi$. Then $\varphi^* : C_p(\mathbb{I}) \to C_p(\mathfrak{c}) = \mathbb{R}^{\mathfrak{c}}$ is a dense embedding. By Theorem 15 there exists a countable dense set $Y \subset C_p(\mathbb{I})$ without non-trivial convergent sequences. Then $\varphi^*(Y)$ is a countable dense subset of $\mathbb{R}^{\mathfrak{c}}$ without non-trivial convergent sequences.

17. Corollary.

It is independent of ZFC whether $\mathbb{R}^{\omega_1}$ has a countable dense subset without non-trivial convergent sequences.

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Proof.

Under CH the space $\mathbb{R}^{\omega_1}$ coincides with $\mathbb{R}^{\mathfrak{c}}$ so it has a countable dense set without non-trivial convergent sequences by Corollary 16. However, if MA$+\neg$CH holds, then every countable subspace of $\mathbb{R}^{\omega_1}$ is Fréchet–Urysohn so it has no countable dense subspace without non-trivial convergent sequences.

18. Corollary.

If X a space with a countable network that has an uncountable compact subspace, then $C_p(X)$ has a dense subset without non-trivial convergent sequences.

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Proof.

If K is an uncountable compact subset of X, then the space K is metrizable and hence $C_p(X)$ is homeomorphic to the product $C_p(K) \times I_K$ where $I_K = \{f \in C_p(X) : f(K) \subset \{0\}\}$. Now it follows $nw(I_K) \leqslant nw(C_p(X)) = nw(X) = \omega$ together with Theorem 15 and Corollary 13 that $C_p(X)$ has a dense subset without non-trivial convergent sequences.

19. Corollary.

If X is an uncountable analytic space, them $C_p(X)$ has a dense subset without non-trivial convergent sequences.

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If X is an uncountable analytic space, them $C_p(X)$ has a dense subset without non-trivial convergent sequences.

Proof.

It is well known that there exists a subspace $K \subset X$ homeomorphic to the Cantor set; Corollary 18 does the rest.

20. Lemma.

Given an infinite monolithic compact space X and a separable closed set $F \subset X$, there exists a discrete set $D \subset (X \backslash F) \times (X \backslash F)$ such that $|D| = w(X)$.

Proof.

There is nothing to prove if $w(X) = \omega$ so we can assume that $\kappa = w(X) > \omega$. Let $\Delta = \{(x, x) : x \in X\}$ be the diagonal of the space X. For every point $z = (x, y) \in (X \backslash F)^2 \backslash \Delta$ we can choose cozero sets $U_z, V_z \subset X \backslash F$ such that $x \in U_x,\ y \in V_z$ and $U_z \cap V_z = \emptyset$; then $z \in W_z = U_z \times V_z \subset (X \backslash F)^2 \backslash \Delta$. Since the family $\mathcal{W} = \{W_z : z \in (X \backslash F)^2 \backslash \Delta\}$ is an open cover of $(X \backslash F)^2 \backslash \Delta$, we can apply Shapirovskii's lemma to find a discrete set $D \subset (X \backslash F)^2 \backslash \Delta$ and a family $\mathcal{W}' \subset \mathcal{W}$ such that $|\mathcal{W}'| \leqslant |D|$ and $(X \backslash F)^2 \backslash \Delta \subset \overline{D} \cup \bigcup \mathcal{W}'$.

If $|D| < \kappa$, then apply monolithity of X to see that $nw(\overline{D}) < \kappa$ and hence we can find a family $\mathcal{W}'' \subset \mathcal{W}$ such that $|\mathcal{W}''| < \kappa$ and $\overline{D} \subset \bigcup \mathcal{W}''$. Then the family $\mathcal{U} = \mathcal{W}' \cup \mathcal{W}''$ has cardinality strictly less than κ and $\bigcup \mathcal{U} = (X \backslash F)^2 \backslash \Delta$. There exists a set $A \subset (X \backslash F)^2 \backslash \Delta$ such that $|A| < \kappa$ and $\mathcal{U} = \{W_z : z \in A\}$.

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The set F being second countable, we can find a countable family $\mathcal{V}$ of cozero subsets of X that T_2-separates the points of F. It is easy to see that the family $\mathcal{H} = \{U_z, V_z : z \in A\} \cup \mathcal{V}$ is T_0-separating in X and $|\mathcal{H}| < \kappa$. It is standard that $w(X) \leqslant |\mathcal{H}| < \kappa$; this contradiction shows that $|D| = \kappa$.

21. Lemma.

Suppose that X is a monolithic compact space. If $F \subset X$ is a separable closed subspace of X, then there exists a family $\{D_n : n \in \mathbb{N}\}$ of discrete subspaces of the set $I_F = \{f \in C_p(X) : f(F) \subset \{0\}\}$ such that $D_n \subset D_{n+1}$ for each $n \in \mathbb{N}$ and $D = \bigcup_{n \in \mathbb{N}} D_n$ is dense in I_F.

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Proof.

Our lemma trivially holds if $w(X) = \omega$ so assume that $\kappa = w(X) > \omega$. For every $r > 0$ let $J_r = I_F \cap C_p(X, [-r, r])$ and apply Lemma 20 to find a discrete set $D \subset (X\backslash F)^2$ such that $|D| = w(X)$. It follows from Lemma 14 that $J_1 \subset \overline{\Omega}$ for some discrete set $\Omega \subset J_2$. Fix a disjoint family $\{U_n : n \in \mathbb{N}\}$ of non-empty open subsets of X such that $\bigcup_{n\in\mathbb{N}} U_n \subset X\backslash F$ and pick a point $x_n \in U_n$ for every $n \in \mathbb{N}$.

The set $\Omega_n = \{n \cdot f : f \in \Omega\} \subset J_{2n}$ is discrete and $J_n \subset \overline{\Omega}_n$ for each $n \in \mathbb{N}$. The set $\bigcup_{n\in\mathbb{N}} J_n$ is easily seen to be dense in I_F and hence the set $\bigcup_{n\in\mathbb{N}} \Omega_n$ is also dense in I_F. For each $n \in \mathbb{N}$ take a function $g_n \in C_p(X, [0, 4^n])$ such that $g_n(x_n) = 4^n$ and $g_n(x) = 0$ for all $x \in X \setminus U_n$; let $E_n = g_n + \Omega_n$ for any $n \in \mathbb{N}$.

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If $n, m \in \mathbb{N}$ and $n < m$, then $g(x_m) \in [4^n - 2n, 4^n + 2n]$ for any $g \in \Omega_n$ and $f(x_m) \in [4^m - 2m, 4^m + 2m]$ for any $f \in \Omega_m$. It is easy to see that $4^m - 2m > 4^n + 2n$ so the intervals $[4^n - 2n, 4^n + 2n]$ and $[4^m - 2m, 4^m + 2m]$ are disjoint and hence

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$(*)$ $\overline{E}_n \cap \overline{E}_m = \emptyset$ for any distinct $n, m \in \mathbb{N}$.

Let $D_n = E_1 \cup \ldots \cup E_n$; then $D_n \subset D_{n+1}$ and it easily follows from $(*)$ that the set D_n is discrete for each $n \in \mathbb{N}$. To see that $D = \bigcup_{n\in\mathbb{N}} D_n = \bigcup_{n\in\mathbb{N}} E_n$ is dense in I_F take any points $y_1, \ldots, y_k \in X \backslash F$ and $O_1, \ldots, O_k \in \tau^*(\mathbb{R})$.

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There exists $n \in \mathbb{N}$ such that $\{y_1, \ldots, y_k\} \cap U_n = \emptyset$ and $O_i \cap [-n, n] \neq \emptyset$ for any $i \leqslant k$. It follows from $J_n \subset \overline{\Omega}_n$ that we can find a function $f \in \Omega_n$ such that $f(y_i) \in O_i$ for all $i \leqslant k$. However, $g_n(y_i) = 0$ and hence $f(y_i) + g_n(y_i) = f(y_i) \in O_i$ for every $i \leqslant k$ so the function $f + g_n \in E_n$ witnesses that D is dense in I_F.

The result that follows shows that, for wide classes of compact spaces X, if $C_p(X)$ is not Fréchet–Urysohn, then it is very strongly non-Fréchet–Urysohn, i.e., there exists an dense subset of $C_p(X)$ that has non non-trivial convergent sequences.

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22. Theorem.

For any monolithic compact space X, the following conditions are equivalent:
(a) $C_p(X)$ is Fréchet–Urysohn;
(b) X is scattered;
(c) every dense set $D \subset C_p(X)$ has a non-trivial convergent sequence.

Proof.

It is well known that (a) $\Longleftrightarrow$ (b) and, trivially, (a)$\Longrightarrow$(c). Now, if X is not scattered, then there exists a separable closed crowded set $F \subset X$. Since F is metrizable, there exists a continuous linear map $\varphi : C_p(F) \to C_p(X)$ such that $\varphi(f)|F = f$ for each $f \in C_p(F)$.

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As an easy consequence, the space $C_p(X)$ is linearly homeomorphic to the product $C_p(F) \times I_F$ where $I_F = \{f \in C_p(X) : f(F) \subset \{0\}\}$. It follows from Theorem 15 that we can apply Proposition 12 and Lemma 21 to see that $C_p(X)$ has a dense subspace without non-trivial convergent sequences; this contradiction with (c) shows that (c)$\Longrightarrow$(b).

23. Corollary.

A monolithic compact space X is not scattered if and only if $C_p(X)$ has a dense subspace without non-trivial convergent sequences.

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25. Corollary.

For any cardinal $\kappa \geqslant \mathfrak{c}$, the space $\mathbb{R}^\kappa$ has a dense subset with no non-trivial convergent sequences.

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Proof.

Let $A(\kappa)$ be the one-point compactification of a discrete space $D(\kappa)$ of cardinality κ. Since $X = A(\kappa) \oplus \mathbb{I}$ is a non-scattered Eberlein compact, we can apply Corollary 24 to see that $C_p(X)$ has a dense subset Y with no non-trivial convergent sequences. The set Y is also dense in $\mathbb{R}^X$; since $|X| = \kappa$, the spaces $\mathbb{R}^X$ and $\mathbb{R}^\kappa$ are homeomorphic so $\mathbb{R}^\kappa$ also has a dense subset without non-trivial convergent sequences.

1. Problem.

Let X be a non-scattered compact space. Is it true that $C_p(X)$ has a dense subset without non-trivial convergent sequences?

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Let X be a non-scattered first countable compact space. Is it true that $C_p(X)$ has a dense subset without non-trivial convergent sequences?

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3. Problem.

Does $C_p(\beta\omega)$ have a dense subset without non-trivial convergent sequences?

4. Problem.

Suppose that X is a crowded linearly ordered compact topological space. Must $C_p(X)$ have a dense subset without non-trivial convergent sequences?

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5. Problem.

Let X be a countably compact space such that $C_p(X)$ has a dense Fréchet–Urysohn subspace. Must X be scattered?

4. Problem.

Suppose that X is a crowded linearly ordered compact topological space. Must $C_p(X)$ have a dense subset without non-trivial convergent sequences?

5. Problem.

Let X be a countably compact space such that $C_p(X)$ has a dense Fréchet–Urysohn subspace. Must X be scattered?

6. Problem.

Let X be a countably compact space of countable tightness such that $C_p(X)$ has a dense Fréchet–Urysohn subspace. Must X be scattered?

7. Problem.

Let X be a pseudocompact space such that $C_p(X)$ has a dense Fréchet–Urysohn subspace. Must X be scattered?

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8. Problem.

Let X be a first countable pseudocompact space such that $C_p(X)$ has a dense Fréchet–Urysohn subspace. Must X be scattered?

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Let X be a first countable pseudocompact space such that $C_p(X)$ has a dense Fréchet–Urysohn subspace. Must X be scattered?

9. Problem.

Let X be a first countable pseudocompact non-scattered space. Must $C_p(X)$ have the Banakh property?

10. Problem.

Let X be a countably compact non-scattered space of countable tightness. Must $C_p(X)$ have the Banakh property?

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Let X be a countably compact non-scattered space of countable tightness. Must $C_p(X)$ have the Banakh property?

11. Problem.

Suppose that X is a scattered space. Must $C_p(X)$ have a dense Fréchet–Urysohn subspace?

10. Problem.

Let X be a countably compact non-scattered space of countable tightness. Must $C_p(X)$ have the Banakh property?

11. Problem.

Suppose that X is a scattered space. Must $C_p(X)$ have a dense Fréchet–Urysohn subspace?

12. Problem.

Suppose that X is a scattered metrizable space. Must $C_p(X)$ have a dense Fréchet–Urysohn subspace?

13. Problem.

Suppose that X is a pseudocompact scattered space. Must $C_p(X)$ have a dense Fréchet–Urysohn subspace?

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Suppose that X is a pseudocompact scattered space. Must $C_p(X)$ have a dense Fréchet–Urysohn subspace?

14. Problem.

Suppose that X is a countably compact scattered space. Must $C_p(X)$ have a dense Fréchet–Urysohn subspace?

15. Problem.

Suppose that X is an infinite compact space. Must $C_p(C_p(X))$ have a dense subspace without non-trivial convergent sequences?

13. Problem.

Suppose that X is a pseudocompact scattered space. Must $C_p(X)$ have a dense Fréchet–Urysohn subspace?

14. Problem.

Suppose that X is a countably compact scattered space. Must $C_p(X)$ have a dense Fréchet–Urysohn subspace?

15. Problem.

Suppose that X is an infinite compact space. Must $C_p(C_p(X))$ have a dense subspace without non-trivial convergent sequences?

16. Problem.

Is it true that $C_p(X)$ has a dense subspace of countable tightness for any space X?

17. Problem.

Is it true that $C_p(C_p(X))$ has a dense subspace of countable tightness for any space X?

17. Problem.

Is it true that $C_p(C_p(X))$ has a dense subspace of countable tightness for any space X?

18. Problem.

Is it true that $C_p(X)$ has a dense subspace of countable tightness for any pseudocompact space X?

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19. Problem.

Is it true that $C_p(X)$ has a dense subspace of countable tightness for any countably compact space X?

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Is it true that $C_p(X)$ has a dense subspace of countable tightness for any countably compact space X?

20. Problem.

Is it true that $C_p(X)$ has a dense subspace of countable tightness for any Lindelöf space X?

Thanks for your attention!!!