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Which Topological Groups Arise as Automorphism Groups of Locally Finite Graphs?

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Which topological groups arise as automorphism groups of locally finite graphs?

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June 2017

Background

- A **topological group** G is a group with Hausdorff topology such that multiplication and inversion are both continuous.
- Let Γ be a locally finite graph and τ_p be the pointwise topology. Then $(\mathrm{Aut}(\Gamma), \tau_p)$ is a topological group.
- G is a **profinite group** if $G \cong \varprojlim F_\lambda$ where F_λ are finite groups with canonical group homomorphisms $\phi_{\lambda,\mu} : F_\lambda \to F_\mu$ if $F_\mu \leq F_\lambda$.
- Profinite group $=$ compact, and 0-dimensional topological group.

Frucht Theorem

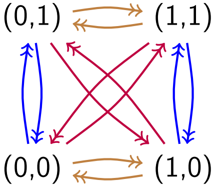
Theorem (Frucht, 1939)

For every **finite** *group F, there exists a finite graph Γ such that $\varphi : Aut(\Gamma) \to F$ is an isomorphism.*

A **rigid graph** is a locally finite graph R such that $\mathrm{Aut}(R) = \{1\}$.

Examples

$\mathbb{Z}_2^2$



$\mathbb{Z}_2$

Problem

What is Aut(Γ)? What kind of topological group is Aut(Γ)?

Know:

- If Γ is a locally finite graph and $v \in \Gamma$, then $\mathrm{Aut}(\Gamma)_v$ is profinite.
- Further if Γ is a locally finite and countable graph, then $\mathrm{Aut}(\Gamma)_v$ is an open subgroup of countable index.

Looking for converse...

Let G be a separable metrizable topological group and let $U \leq G$ be an open profinite subgroup.

Does there exists a countable locally finite graph Γ such that

$$Aut(\Gamma) \cong G?$$

Answer: YES!

Theorem

Let G be a separable metrizable topological group with an open profinite subgroup. Then $\exists$ a countable, connected, locally finite graph Γ_G such that

$$Aut(\Gamma) \cong G.$$

Corollary

For every countable group G, $\exists$ a countable, connected, locally finite graph Γ s.t.

$$\mathrm{Aut}(\Gamma) \cong G.$$

Theorem

A group G is topologically isomorphic to a separable metrizble topological group with an open profinite subgroup iff $G \cong Aut(\Gamma)$ for some connected, locally finite graph.

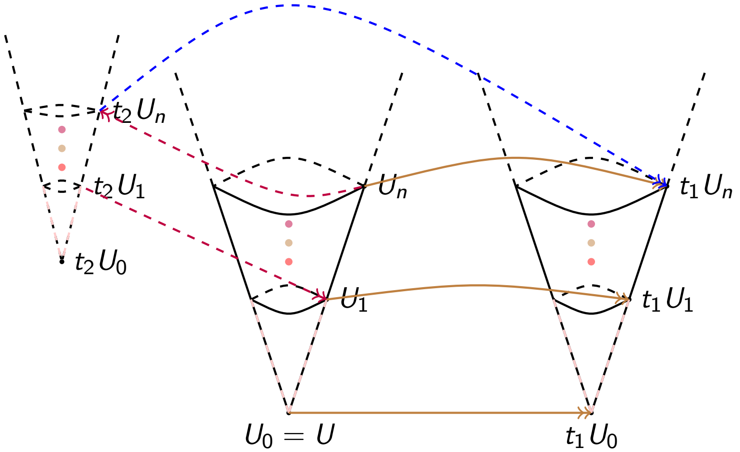
Proof Strategy

1. Construct a colored and directed, countably, locally finite graph $\mathcal{C}$.
2. Verify $\mathrm{Aut}(\mathcal{C}) \cong G$.
3. Replace the colored directed edges by rigid graphs to obtain Γ.
4. Verify that $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathcal{C})$.

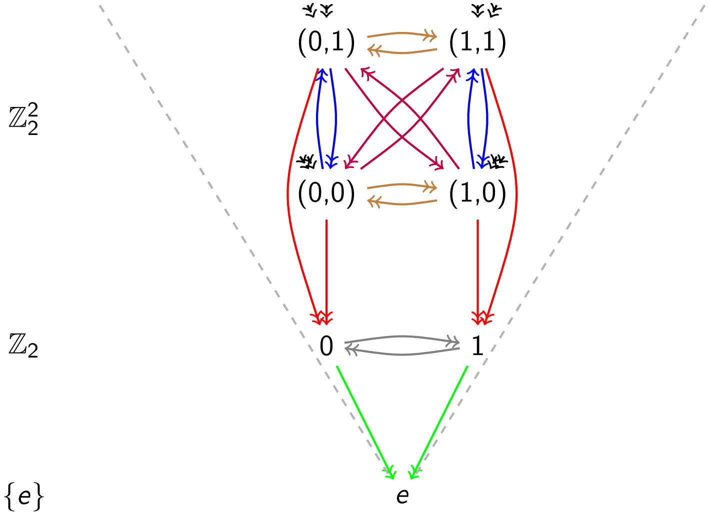
Part 1 - Construct $\mathcal{C}$ (Ingredients)

- $U \leq G$ open, profinite and metrizable.
  $U_0 = U \supseteq U_1 \supseteq \cdots \supseteq U_n \supseteq \dots$ nested sequence of open subgroups of U and $\bigcap_n U_n = \{1\}$.
- Countable dense $T = \{t_0, \dots, t_n, \dots\}$.
  Finite subset $T_n = \{t_0, \dots, t_n\}$ with coloring function $c_n : T_n \to 2\mathbb{N}$ such that the colors are distinct between levels.
- Countably many levels of $\mathcal{C}_n = G/U_n$ and $\mathcal{C}_{\leq n} = \bigcup_{m \leq n} \mathcal{C}_m$.
- $\mathcal{C} = \bigcup_n \mathcal{C}_n$.
- 'Horizontal' edge from hU_n to htU_n with color $c_n(t)$ for $t \in T_n$.
  'Vertical' edge from $h'U_{n+1}$ to hU_n s.t. $hU_n \supseteq h'U_{n+1}$ with color $2n+1$.

Picture Time



Picture - Zoom in



Part 1 - Construct $\mathcal{C}$ (Key Properties)

- $\mathcal{C}$ is countable and locally finite and *connected*.
- For each n, $\mathcal{C}_n$ and $\mathcal{C}_{\leq n}$ are invariant under $\text{Aut}(\mathcal{C})$.
- If $\alpha : \mathcal{C} \to \mathcal{C}$ such that $\alpha|_{\mathcal{C}_{\leq n}}$ is an automorphism for all n, then $\alpha \in \text{Aut}(\mathcal{C})$.

Part 2 - $\text{Aut}(\mathcal{C}) \cong G$ (Definition)

Define $\text{pt}(\{g\}) = g$.

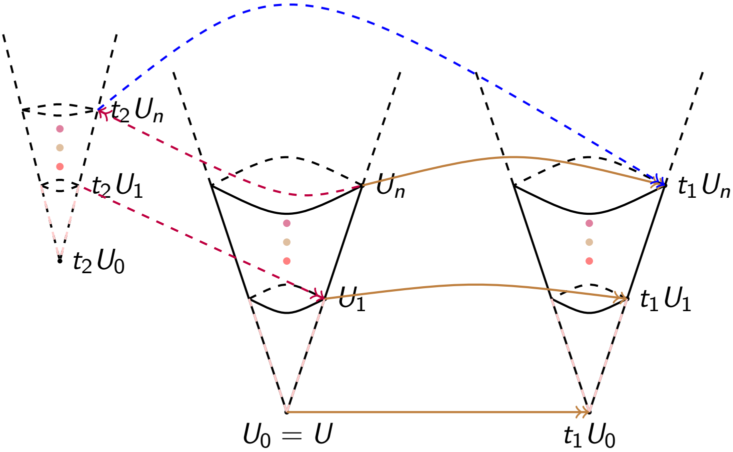
Define the topological isomorphisms

$$\Psi : G \to \text{Aut}(\mathcal{C}) \qquad \text{by } \Psi(g)(htU_n) = ghtU_n$$

$$\Phi : \text{Aut}(\mathcal{C}) \to G \qquad \text{by } \Phi(\alpha) = \text{pt}\left(\bigcap_n \alpha(U_n)\right)$$

Part 2 - $\mathrm{Aut}(\mathcal{C}) \cong G$ (Topological isomorphism)

$G, \mathrm{Aut}(\mathcal{C})$ both have topology τ_p.



Part 3 - Construct Γ

- Enumerate finite, connected, rigid graphs $\{R_n : n \in \mathbb{N}\}$ such that $|R_n| > |R_m| > 2$ iff $n > m$.
- Replace colored-edges by isomorphism types of R_n.

Call the resulting graph Γ.

It has the same key properties as $\mathcal{C}$.
Hence $\mathrm{Aut}(\Gamma) \cong \mathrm{Aut}(\mathcal{C})(\cong G)$.

What's Next?

Construct a countable, connected, **non-locally finite** graph Γ such that $\mathrm{Aut}(\Gamma)$ is **topologically isomorphic** to a given Polish non-archimedean group.

Thank you