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Truncation Error for a Finite Difference Scheme for the Black-Scholes Model

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ABSTRACT

Finite difference methods are simplest and oldest methods among all the numerical techniques to approximate the solution of partial differential equations (PDEs). The derivatives in the partial differential equations are approximated by finite difference formula. The error between the numerical solution and the exact solution is determined by the error between a differential operator to a difference operator. This error is called the discretization error or truncation error. The truncation error reflects the fact that a finite part of a Taylor series is used in the approximations. In this work we will analyze the truncation error for a finite difference scheme for the Black-Scholes PDE for the valuation of an option.

1 Introduction

The Black-Scholes (BS) Option Pricing Model is one of the prominent models in finance. It was published by Fischer Black and Myron Scholes in their 1973 paper, The Pricing of Options and Corporate Liabilities. Their work involved calculating a derivative to measure how the discount rate of a warrant varies with time and stock price. The result was a partial differential equation similar to the heat transfer equation. The standard BS-PDE is

\[
\frac{\partial C}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0
\]

where

\(C(S,t)\) is the price of a European option,
\(S\) is the underlying asset price (e.g. stock price, an index),
\(t\) is the time to expiration,
\(r\) is the interest rate,
\(\sigma\) is the continuous-compounding risk-free rate (annual),
\(D\) is the volatility of the underlying asset.

The formula below is an example of a simple option pricing method calculated using the analytical solution of the Black-Scholes formula with changing stock prices and time.

\[
\text{Price} = \text{max}(S - K, 0) e^{-rt} + \frac{1}{2} \left[ N\left(\frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2} \sigma^2\right)T}{\sigma \sqrt{T}}\right) - N\left(\frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2} \sigma^2\right)T}{\sigma \sqrt{T}}\right)\right]
\]

This formula is used to price plain vanilla European call options. The formula does not work well in pricing more complex exotic options. Most of these options are path dependent and there is no closed form solution for the price. Hence, the BS formula is not useful in pricing complex options.

Therefore, our motivation is to solve the BS-PDE numerically with different boundary conditions as most exotic options have different conditions to those used in solving the BS-PDE for simple European call options. First step towards numerical solution of partial differential equations is the discretization of PDE. By discretization we mean any method of reducing continuous PDE to a discrete set of difference equations that can be solved on a computer. We discretize the PDE and obtain the truncation error of the finite difference schemes.

Truncation error is the error made by cutting off an infinite sum and approximating it by a finite sum. We take a finite number of steps to approximate the solution to the BS-PDE.

2 Black-Scholes Partial Differential Equation

We begin writing our general heat equation

\[
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}
\]

We rewrite the coefficients of the differential operators by replacing them with letters \(A\), \(B\), and \(D\)

\[
\frac{\partial u}{\partial t} = \frac{1}{2} \sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + (r - \frac{1}{2} \sigma^2) \frac{\partial u}{\partial x}
\]

To solve the PDE numerically we need to discretize the PDE (1). We construct a weighted finite difference approximation to each of the derivative terms in the PDE. We use weighted finite difference scheme with \(\alpha\) a temporal weight and \(\beta\) a spatial weight.

We get a long form of \(\psi_{n+1}^j\) and \(\psi_{n}^j\) which we then substitute in equation (2) to get a discretized PDE

\[
\frac{\psi_{n+1}^j - \psi_{n}^j}{\Delta t} = \frac{1}{2} \sigma^2 x^2 \left[ \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x^2} + \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x^2} \right] + \frac{r - \frac{1}{2} \sigma^2}{\Delta x} \left[ \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x} \right] - r \psi_{n}^j
\]

\[
\left( \frac{\psi_{n+1}^j - \psi_{n}^j}{\Delta t} \right) = \frac{1}{2} \sigma^2 x^2 \left[ \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x^2} + \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x^2} \right] + \frac{r - \frac{1}{2} \sigma^2}{\Delta x} \left[ \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x} \right] - r \psi_{n}^j
\]

We again substitute this and using the following cross partial derivatives in (4) and (7)

\[
\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial t} \right]
\]

\[
\frac{\partial^2 u}{\partial x \partial t} = \frac{\partial}{\partial x} \left[ \frac{\partial u}{\partial t} \right]
\]

Now, we can compare equations (4) and (7). In (7) the coefficients of the partial derivatives correspond to the exact coefficients in (4). We call these numerical coefficients and the difference between these coefficients and the exact ones is the truncation error. The form of PDE [4] with numerical coefficients is

\[
\frac{\psi_{n+1}^j - \psi_{n}^j}{\Delta t} = A \psi_{n+1}^j + B \psi_{n}^j + C \psi_{n+1}^j + D \psi_{n}^j
\]

where "num" stands for numerical coefficients.

<table>
<thead>
<tr>
<th>(t_n)</th>
<th>(t_{n+1})</th>
<th>(t_{n+2})</th>
<th>(t_{n+3})</th>
<th>(t_{n+4})</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A)</td>
<td>(B)</td>
<td>(C)</td>
<td>(D)</td>
<td></td>
</tr>
</tbody>
</table>

Now, we have two truncated PDEs. If we substitute \(\psi_{n+1}^j = \psi_{n}^j\) and \(\psi_{n+1}^j = \psi_{n}^j\) in equation (8) we obtain the Crank-Nicolson approximation of the truncation error in \(A\), \(B\) and \(D\)

\[
t_2 = \frac{1}{2} \sigma^2 x^2 \left[ \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x^2} + \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x^2} \right] + \frac{r - \frac{1}{2} \sigma^2}{\Delta x} \left[ \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x} \right] - r \psi_{n}^j
\]

\[
t_3 = \frac{1}{2} \sigma^2 x^2 \left[ \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x^2} + \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x^2} \right] + \frac{r - \frac{1}{2} \sigma^2}{\Delta x} \left[ \frac{\psi_{n+1}^{j+1} - 2\psi_{n+1}^j + \psi_{n+1}^{j-1}}{\Delta x} \right] - r \psi_{n}^j
\]

We have demonstrated how to apply the finite difference method to approximate the solution of the BS-PDE. Apart from the truncation error analysis, we have found the truncation error formula in the numerical coefficients. Using this approach we can improve approximations of the BS-PDE by carefully selecting boundary conditions that will minimize the errors. The finite difference method can be used to price more complex, exotic, options which we could not solve exactly. In our future work we plan to use the corrected coefficients instead of \(A_{num}\), \(B_{num}\) and \(D_{num}\) and simulate the numerical solution of the scheme for the valuation of different options with appropriate boundary conditions.

References

1. Figures 1, 2. http://www.math.uchicago.edu/\~vojta/cheb/PHYP/Box.png


